1. FINANCIAL MARKETS IN FINITE DISCRETE TIME

- PROBABILITY SPACE (Ω, F, P)
- FILTRATION $\mathbb{F} = (F_k)_{k=0,1,2\dots T}$

  $\mathbb{F}$ is a family of σ-fields $\underbrace{(F_k \subseteq F_\ell \quad k \leq \ell)}_{\text{all events observable up to } K}$

- $X = (X_k)_{k=0,1\dots T}$ STOCHASTIC process
- X is ADAPTED TO $\mathbb{F}$ if each X_k is F_k-measurable
- X is PREDICTABLE if each X_k is F_{k-1}-measurable

EXAMPLE: MULTIPLICATIVE MODEL

$$\tilde{S}^0_k := \prod_{j=1}^{k} (1+r_j)$$ BANK ACCOUNT WITH IR r_j $\qquad r_k > -1$

$$\tilde{S}^1_k = S^1_0 \prod_{j=1}^{k} Y_j$$ STOCK WITH GROWTH FACTOR Y_j $\qquad Y_k > 0 \qquad S^1_0 > 0$

The most common choice for filtration is:

$$F_k = \sigma(Y_1, Y_2 \dots Y_k) = \sigma(\tilde{S}^1_0, \tilde{S}^1_1 \dots \tilde{S}^1_k)$$

- $\tilde{S}^1$ is ADAPTED to $\mathbb{F}$
- $\tilde{S}^0$ is PREDICTABLE (r_k is usually known at $k-1$)

BINOMIAL MODEL

$r_k = r > -1 \quad \forall k$

$$\frac{\tilde{S}^1_k}{\tilde{S}^1_{k-1}} = Y_k = \begin{cases} 1+u & \text{prob. } p \\ 1+d & \text{prob } (1-p) \end{cases}$$

COX-ROSS-RUBINSTEIN (CRR) BIN. MODEL

$$S^1_k := \frac{\tilde{S}^1_k}{\tilde{S}^0_k}$$ DISCOUNTED ASSET PRICE

$$S^0_k = \frac{\tilde{S}^0_k}{\tilde{S}^0_k} = 1 \quad \forall k$$ equivalent of working with $\emptyset$ (ZERO) INTEREST

S^0 : reference asset or numeraire
S^1 : risky assets with d components

$\Big\}$ $d+1$ TRADABLE ASSETS

TRADING STRATEGY

It is a $\mathbb{R}^{d+1}$-valued stochastic process $\varphi = (\varphi^0, \theta)$ where $\varphi^0 = (\varphi^0_k)_{k=0..T}$ is real-valued and adapted and $\theta = (\theta_k)$ with $\theta_0 = \emptyset$ is real-valued and predictable.

The (discounted) value process is:

$$V_k(\varphi) := \varphi^0_k S^0_k + \theta^T_k S_k = \varphi^0_k + \sum_{i=1}^{d} \theta^i_k S^i_k$$

It describes a portfolio evolving in $d+1$ assets, with φ^0_k units in the bank account and θ^i_k shares of stock i.

Note that: • θ_k is determined at time $k-1$!
• φ^0_k is determined at time k

V_k is the value of the strategy at time k before the trade.

For every stochastic process we can define:

$$\Delta X_k := X_k - X_{k-1}$$

INCREMENTAL AND TOTAL COST

$$\boxed{\Delta C_{k+1}(\varphi) = C_{k+1}(\varphi) - C_k(\varphi)}$$

$$= \left(\varphi^0_{k+1} - \varphi^0_k\right) S^0_k + \left(\theta_{k+1} - \theta_k\right)^T S_k$$

$$= \varphi^0_{k+1} - \varphi^0_k + \sum_{i=1}^{d} \left(\theta^i_{k+1} - \theta^i_k\right) S^i_k$$

$$= \Delta V_{k+1}(\varphi) - \theta^T_{k+1} \Delta S_{k+1}$$

$$\boxed{C_k(\varphi) = C_0(\varphi) + \sum_{j=1}^{k} \Delta C_j(\varphi)}$$

Cumulative total cost

$$= V_k(\varphi) - G_k(\theta)$$

GAIN PROCESS

(DISCOUNTED) GAINS

The discounted gain process associated with φ is:

$$\boxed{G_K(\theta) := \sum_{j=1}^{K} \theta_j^T \, \Delta S_j}$$

SELF-FINANCING

φ is SELF-FINANCING if $C(\varphi)$ is COSTANT OVERTIME

$$\left[C(\varphi) = C_0(\varphi) = V_0(\varphi) = \varphi_0^0 \right]$$

another equivalent condition: $V(\varphi) = V_0(\varphi) + G(\theta) = \varphi_0^0 + G(\theta)$

PROPOSITION

If φ SELF-FINANCING, then (φ_K^0) is PREDICTABLE

(any self-financing strategy is uniquely determined by V^0, θ)

STOPPING TIMES

τ is a stopping time if, at any time j you can answer YES/NO at "Has event τ happened yet?" using only info up to time j.

A random variable τ is a stopping time if, for every K,

$$\boxed{\{\tau \leq K\} \in \mathcal{F}_K}$$

(equivalent to: $\{\tau = K\} \in \mathcal{F}_K$)

ADMISSIBLE STRATEGY

φ is a-admissible if $V_K(\varphi) \geq -a$ P-a.s. for every K, $a \geq 0$

φ is admissible if it is a-admissible for some $a \geq 0$.

Intuition: an admissible trading strategy has a credit line with a lower bound.

MARTINGALES

$(\Omega, \mathcal{F}, \mathbb{Q})$ with filtration $\mathbb{F} = (\mathcal{F}_k)_{\dots}$

$X = (X_k)_{\dots}$ is a martingale (w.r.t. $\mathbb{Q}$ and $\mathbb{F}$) if

- adapted to $\mathbb{F}$
- $\mathbb{Q}$-integrable
- satisfies martingale property: $\mathbb{E}_{\mathbb{Q}}[X_\ell | \mathcal{F}_k] = X_k$ for $\ell \geq k$

if $\mathbb{E}_{\mathbb{Q}}[X_\ell | \mathcal{F}_k] \geq X_k$: SUBMARTINGALE (Tendency to go down)

if $\mathbb{E}_{\mathbb{Q}}[X_\ell | \mathcal{F}_k] \leq X_k$ SUPERMARTINGALE

EXAMPLE: BINOMIAL MODEL

Consider Binomial Model. S^1 is a martingale if:

$$\mathbb{E}_{\mathbb{Q}}\left[S^1_{k+1} | \mathcal{F}_k\right] = S_k$$

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{S^1_{k+1}}{S^1_k} \middle| \mathcal{F}_k\right] = 1$$

$$\left.\begin{aligned} S^1_{k+1} &= \frac{\tilde{S}^1_{k+1}}{\tilde{S}^0_{k+1}} = \frac{S^1_0 \prod^{k+1} Y_j}{\prod^{k+1} 1+r} \\ S^1_k &= \frac{\tilde{S}^1_{k+1}}{\tilde{S}^0_k} = \frac{S^1_0 \prod^{k} Y_j}{\prod^{k} 1+r} \end{aligned}\right\} \frac{S^1_{k+1}}{S^1_k} = \frac{Y_{k+1}}{1+r}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{Y_{k+1}}{1+r} \middle| \mathcal{F}_k\right] = \frac{p(1+u) + (1+d)(1-p)}{1+r} = 1$$

$$p(1+u) + (1+d)(1-p) = 1 + r \longrightarrow \boxed{r = pu + (1-p)d}$$

LOCAL MARTINGALE

X is a local martingale if there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times s.t. for each $n \in \mathbb{N}$, $X^{\tau_n} = (X_{\tau_n \wedge k})_{k=\dots}$ is a martingale.

(τ_n) is then a STOPPING SEQUENCE

THEOREM

Suppose X is (local) martingale null at zero.
For any predictable θ, the stochastic integral process

$$\theta \cdot X = \sum_{j=1}^{k} \theta_j^T \Delta X_j$$ is a LOCAL MARTINGALE null at zero.

If θ bounded, it's a MARTINGALE.

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2. ARBITRAGE AND MARTINGALE MEASURES

ARBITRAGE OPPORTUNITY

It is a self-financing strategy $\varphi \hat{=} (0, \theta)$ with ZERO INITIAL WEALTH with $V_T(\varphi) \geq 0$ P-a.s. and $P[V_T(\varphi) > 0] > 0$.

It produces SOMETHING from NOTHING without ANY RISK!

- $P(V_T(\varphi) > 0) > 0$ FIRST KIND
- $P(V_0(\varphi) < 0) > 0$ SECOND KIND

- NA: impossible to produce profits with an admissible φ
- NA_+: impossible to produce profits with 0-admissible φ
- NA': impossible to produce profits with any self financing strategy

In discrete time:

$NA \Longleftrightarrow NA_+ \Longleftrightarrow NA'$

In general

$NA' \Longrightarrow NA \Longrightarrow NA_+$

PROPOSITION: DISCRETE TIME & NA

In discrete time, the following are equivalent:

1. Financial market S arbitrage free
2. There exists NO φ with zero initial wealth satisfying $P(V_T(\varphi) > 0) > 0$ and $V_T(\varphi) \geq 0$ P-a.s. ($\approx S$ satisfies NA')
3. For every s-financing φ with $V_0(\varphi) = 0$ and $V_T(\varphi) \geq 0$ P-a.s., we have $V_T(\varphi) = 0$
4. For the space:

$$G' := \{V_T(\varphi) : \varphi = \{G_T(\theta) : \theta \text{ is } \mathbb{R}^d\text{-valued and predictable}\}$$ we have $G' \cap L^0_+(\mathcal{F}_T) = \{0\}$

- measurability
- non-negativity

LEMMA

If there exists a prob. measure $\mathbb{Q} \approx \mathbb{P}$ st S is a $\mathbb{Q}$-mart. Then S is arbitrage free.

COROLLARY

In the multinomial model with $y_1 < \dots < y_m$ and r, there exists $\mathbb{Q} \approx \mathbb{P}$ st $\tilde{S}^1/\tilde{S}^0$ is a $\mathbb{Q}$-martingale iff $y_1 < r < y_m$

In particular, in the Binomial Model, we have:

$u > r > d$

$$\boxed{\mathbb{Q}[Y_K = 1+u] = q^* = \frac{r-d}{u-d} = 1 - \mathbb{Q}[Y_K = 1+d]}$$

EQUIVALENT (LOCAL) MARTINGALE MEASURE

A E(L)MM for S is a prob. measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ such that S is a (local) $\mathbb{Q}$-martingale

- $\mathbb{P}_e(S)$: set of all EMMs for S
- $\mathbb{P}_{e,loc}(S)$: set of all ELMMs for S

$\mathbb{P}_e \subseteq \mathbb{P}_{e,loc}$ because every martingale is also a local martingale

FOUNDAMENTAL THEOREM OF ASSET PRICING (FTAP) [DALANG, MORTON, WILLINGER (DMW)]

NA for S with $S_0 = 1$ $\longleftrightarrow$ $\mathbb{P}_{e(S)} \neq \emptyset$ $\longleftrightarrow$ $\mathbb{P}_{e,loc(S)} \neq \emptyset$

EQUIVALENT MARTINGALE MEASURES

$\mathbb{Q}, \mathbb{P}$ probability measures, $\mathbb{Q} \approx \mathbb{P}$

$$\boxed{\frac{d\mathbb{Q}}{d\mathbb{P}} = D \quad \text{random variable st} \quad \mathbb{Q}[A] = \mathbb{E}_P[D I_A] \quad \mathbb{E}_Q[Y] = \mathbb{E}_P[YD] \ \forall Y \geq 0}$$

(note that $\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{D}$)

$$\boxed{Z_K := \mathbb{E}_P[D \mid \mathcal{F}_K] = \mathbb{E}_P\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_K\right]}$$ is a P-martingale

LEMMA

- $\forall K = 0, 1 \dots T$ and any $A \in \mathcal{F}_K$ or any $\mathcal{F}_K$-measurable rand. var $Y \geq 0$:

$$\boxed{\mathbb{Q}[A] = \mathbb{E}_P[Z_K I_A] \quad \text{and} \quad \mathbb{E}_Q[Y] = \mathbb{E}_P[Z_K Y]}$$

Z_K is the density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ on $\mathcal{F}_K$

- if $J \leq K$ and U_K $\mathcal{F}_K$-measurable and $U_K \geq 0$ or $\in L^1(\mathbb{Q})$:

$$\boxed{\mathbb{E}_Q[U_K \mid \mathcal{F}_J] = \frac{1}{Z_J} \mathbb{E}_P[Z_K U_K \mid \mathcal{F}_J] \quad \text{BAYES FORMULA}} \quad \begin{array}{l} Q\text{-a.s.} \\ P\text{-a.s} \end{array}$$

- Process N is adapted to $\mathbb{F}$ if it's a $\mathbb{Q}$-martingale and if NZ is a $\mathbb{P}$-martingale

EXAMPLE (HOW TO CONSTRUCT $\mathbb{Q}$)

$$1 = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_k^1}{S_{k-1}^1} \,\Big|\, \mathcal{F}_{k-1}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_k}{1+r} \,\Big|\, \mathcal{F}_{k-1}\right] = \mathbb{E}_P\left[\frac{D_k Y_k}{1+r} \,\Big|\, \mathcal{F}_{k-1}\right]$$

To simplify we set $D_k = g_k(Y_k)$ and

$$\mathbb{E}_P\left[g_k(Y_k)\right] = 1$$ (it still has to be a probability measure)

$$\mathbb{E}\left[D_k Y_k \,\big|\, \mathcal{F}_{k-1}\right] = \mathbb{E}_P\left[Y_k\, g_k(Y_k)\right] = 1+r$$ (it still has to be a martingale)

Now we assume:

Y_k random variables taking values $(1+y_j)_{j\in N}$

$$\mathbb{P}\left[Y_k = 1+y_i\right] = p_i$$

Then g_1 is determined by $g_1(1+y_1)$ and we need

$$q_j = p_j\, g_1(1+y_1)$$

To find $\mathbb{Q} \approx \mathbb{P}$

$$1 = \mathbb{E}_P\left[g_1(Y_1)\right] = \sum_{j\in\mathbb{N}} p_j\, g_1(1+y_i) = \sum_{j\in\mathbb{N}} q_j$$

$$1+r = \mathbb{E}_{\mathbb{Q}}\left[Y_1 \underbrace{g_1(Y_1)}_{D_k}\right] = \sum_j p_j(1+y_j)\, g_1(1+y_i) =$$

$$= \sum_{j\in\mathbb{N}} q_j(1+y_i) = 1 + \sum_{j\in\mathbb{N}} q_j y_j$$

or, equivalently: $\sum_{j\in\mathbb{N}} q_j y_i = r$

3. VALUATION AND HEDGING IN COMPLETE MARKETS

EUROPEAN OPTION

A general european option (or payoff) is a rand. var. $H \in L^0_+(\mathcal{F}_T)$

($H \geq 0$, $\mathcal{F}_T$- measurable)

CALL OPTION

An european call option on asset i with maturity T, strike K and volume y is the right (but not the obligation) to buy y units of asset i at price K on time T (independenty of the actual asset price S^i_T)

$$\boxed{H(\omega) = \max\left[0,\ y\left(S^i_T(\omega) - K\right)\right] = y\left[S^i_T(\omega) - K\right]^+}$$

$H = h(S^i_T)$ with the payoff function $h(x) = y(x-K)^+$

ATTAINABLE PAYOFF

A payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable if there exists an admissible self-financing strategy φ with $V_T(\varphi) = H$.

The strategy is said to replicate H (replicating strategy for H)

THEOREM: ARBITRAGE-FREE VALUATION OF ATTAINABLE PAYOFFS

$\mathcal{F}_0 = \{\emptyset, \Omega\}$
we know nothing at time zero

Consider S arbitrage free and $\mathcal{F}_0$ trivial.

Then every attainable H has a unique price process $V^H = (V^H_k)_{k=0,1\ldots}$ which admits NO arbitrage

$$V^H_k = \underbrace{E_Q\left[H \mid \mathcal{F}_k\right]}_{\text{PRICING FORMULA: expected value of } H} = \underbrace{V_k(V_0, \theta) = V_0 + G_k(\theta)}_{\text{value of the replicating strategy}}$$

THEOREM: CHARACTERIZATION OF ATTAINABLE PAYOFFS

Consider discounted market S with $\mathcal{F}_0$ trivial. For any $H \in L^0_+(\mathcal{F}_T)$

In discrete time, the following are equivalent:

1. H attainable
2. $\sup_{Q \in \mathbb{P}_{e,loc}} \mathbb{E}_Q[H] < \infty$ is attained in some $Q^* \in \mathbb{P}_{e,loc(S)}$
3. The mapping $\mathbb{P}_{e,loc(S)} \to \mathbb{R}$, $Q \to \mathbb{E}_Q[H]$ is CONSTANT

COMPLETE MARKETS

A financial market is complete if every payoff $H \in L^0_+(\mathcal{F}_T)$ is attainable

THEOREM: VALUATION AND HEDGING IN COMPLETE MARKETS

Consider S complete and arbitrage free, $\mathcal{F}_0$ trivial.

Then $\forall H \in L^0_+(\mathcal{F}_T)$ there is a unique price process V^H which admits NO arbitrage:

$$V_k^H = \mathbb{E}_Q[H \mid \mathcal{F}_k] = V_k(V_0, \vartheta)$$

THEOREM (COMPLETE MARKETS) [SECOND FOUNDAMENTAL THEOREM OF A. PRICING]

S arbitrage free, $\mathcal{F}_0$ trivial and $\mathcal{F}_T = \mathcal{F}$

NA + completeness $\Longleftrightarrow$ $\#(\mathbb{P}_{e,loc(S)}) = 1$ ($\mathbb{P}_{e,loc(S)}$ singleton)

EXAMPLE: BINOMIAL MODEL

$u > r > d$

$\tilde{S}^0_k = (1+r)^k$

$\tilde{S}^1_k = \tilde{S}^1_0 \prod_{j=1}^{k} Y_j \qquad Y_i = \begin{cases} u & \text{probability } p \\ d & \text{probability } (1-p) \end{cases}$

From previous Theorems we know that any $H \in L^0_+(\mathcal{F}_T)$ is ATTAINABLE with

$V^H_k = \mathbb{E}_{Q^*}[H|\mathcal{F}_k]$ and $\mathbb{Q}^*$ is the unique EMM:

$$\mathbb{Q}^*[Y_k = 1+u] = q^* = \frac{r-d}{u-d} \in (0,1)$$

COROLLARY (PRICE PROCESS V^H_k FOR BINOMIAL MODEL)

$$\tilde{V}^{\tilde{H}}_k = \tilde{S}^0_k \mathbb{E}_{Q^*}\left[\frac{\tilde{H}}{\tilde{S}^0_T}\middle|\mathcal{F}_k\right] = \mathbb{E}_{Q^*}\left[\frac{\tilde{H}\tilde{S}^0_k}{\tilde{S}^0_T}\middle|\mathcal{F}_k\right] = \frac{\tilde{S}^0_k}{\tilde{S}^0_T}\mathbb{E}_{Q^*}\left[\tilde{H}\middle|\mathcal{F}_k\right]$$

EXAMPLE: EUROPEAN CALL

$$\tilde{H} = [\tilde{S}^1_T - \tilde{K}]^+ = [\tilde{S}^1_T - \tilde{K}]\,\mathbb{I}_{\{\tilde{S}^1_T > \tilde{K}\}}$$

$$\{\tilde{S}^1_T > \tilde{K}\} = \left\{\tilde{S}^1_k \prod_{j=k+1}^{T} Y_j > \tilde{K}\right\} = \left\{\sum_{j=k+1}^{T} \log(Y_j) > \log\left(\frac{\tilde{K}}{S^1_k}\right)\right\}$$

now we define $W_1 \dots W_T$ independent so that their sum follows bin. distr.

$$W_j := \mathbb{I}_{\{Y_i = 1+u\}} = \begin{cases} 1 & \text{if } Y_i = 1+u \\ 0 & \text{if } Y_i = 1+d \end{cases}$$

Then:

$$\log(Y_j) = W_j \log(1+u) + (1-W_j)\log(1+d) = \overbrace{W_j \log\left(\frac{1+u}{1+d}\right) + \log(1+d)}^{A}$$

$$\sum_{j=k+1}^{T} \log(Y_j) =: \sum A = W_{k,T} \log\left(\frac{1+u}{1+d}\right) + (T-k)\log(1+d)$$

$$\hookrightarrow W_{k,T} = \sum_{j=k+1}^{T} W_j \sim Bin(T-k, q^*)$$

$$\{S^1_T > \tilde{K}\} = \left\{ W_{T,K} \log\left(\frac{1+u}{1+d}\right) + (T-K)\log(1+d) > \log\left(\frac{\tilde{K}}{\tilde{S}^0_K}\right) \right\}$$

$$= \left\{ W_{T,K} \log\left(\frac{1+u}{1+d}\right) > \log\left(\frac{\tilde{K}}{\tilde{S}^0_K}\right) - (T-K)\log(1+d) \right\}$$

If we now take probability $\mathbb{Q}^*$:

$$\mathbb{Q}^* = \left[\tilde{S}^1_T > \tilde{K} \mid F_K\right] = \mathbb{Q}^*\left[W_{K,T} > \frac{\log\left(\frac{\tilde{K}}{s}\right) - (T-K)\log(1+d)}{\log\left(\frac{1+u}{1+d}\right)} \right]\Bigg|_{s = \tilde{S}^1_K}$$

$$\mathbb{E}_{\mathbb{Q}^*}\left[\tilde{H} \mid F_K\right] = \underbrace{\mathbb{E}_{\mathbb{Q}^*}\left[\tilde{S}^1_T \, \mathbb{I}_{\{S^1_T > K\}}\right]}_{\text{asset part of the option, what we receive}} - \underbrace{K\,\mathbb{E}_{\mathbb{Q}}\left[\tilde{S}^1_T > \tilde{K} \mid F_K\right]}_{\text{cash we pay to exercise the option}}$$

After some (long) calculations, we end up with:

$$\boxed{\begin{aligned} \tilde{V}^H_K &= \tilde{S}^1_K \, \mathbb{Q}^{**}\left[W_{K,T} > K\right] - \tilde{K}\,\frac{\tilde{S}^0_K}{\tilde{S}^0_T}\,\mathbb{Q}^*\left[W_{K,T} > X\right] \\ X &= \frac{\log\left(\frac{\tilde{K}}{s}\right) - (T-K)\log(1+d)}{\log\left(\frac{1+u}{1+d}\right)} \end{aligned}}$$

BINOMIAL PRICING FORMULA IN DISCRETE TIME

$\mathbb{Q}^{**}$ dual martingale measure

4. Basics about Brownian Motion

$(\Omega, F, \mathbb{P})$

$\mathbb{F} = (F_t)$ filtration in countinuos time

$$F_\infty = \bigvee_{t \geq 0} F_t := \sigma\left(\bigcup_{t\geq 0} F_t\right)$$

Brownian Motion

A Brownian motion w.r.t. $\mathbb{P}$ and $\mathbb{F} = F_t$ is a stochastic process $W = (W_t)_{t \geq 0}$ null at zero and adapted to $\mathbb{F}$ that satisfies:

(BM1) For $s \leq t$, the increment $W_t - W_s$ is independent of F_s and follows $\sim N(0, t-s)$

(BM2) W has continuous trajectories. $t \to W(t)$ is continuous

Proposition: Properties of BM

1. $W^1 := -W$ is a BM
2. $W^2_t := W_{t+T} - W_T$ is a BM $t, T > 0$ (restarting at a fixed time)
3. $W^3_t := c W_{\frac{t}{c^2}}$ is a BM, $c \neq 0$ (rescaling in space/time)
4. $W^4_t := W_{T-t} - W_T$ is a BM on $[0, T]$, $0 \leq t \leq T$ (time reversal)
5. W^5_t is a BM

$$W^5_t := \begin{cases} t\, W_{\frac{1}{t}} & t > 0 \\ 0 & t = 0 \end{cases}$$

PROPOSITION: TRAJECTORIES OF BM

1. $\lim_{t\to\infty}\left(\frac{W_t}{t}\right)=0$ LAW OF LARGE NUMBERS

2. With $\psi_{glob}(t):=\sqrt{2t\log(\log(t))}$

$$\left.\begin{array}{l}\limsup_{t\to\infty}\\ \liminf_{t\to\infty}\end{array}\right\}\frac{W_t}{\psi_{glob}(t)}=\begin{cases}+1 & P\text{-a.s.}\\ -1\end{cases}$$

(W_t oscillates between $\pm\psi_{glob}(t)$)

GLOBAL
LAW OF ITERATED LOGARITHM (LIL)

3. With $\psi_{loc}(h)=\sqrt{2h\log(\log(\frac{1}{h}))}$

$$\left.\begin{array}{l}\limsup_{h\to 0}\\ \liminf_{h\to 0}\end{array}\right\}\frac{W_{t+h}-W_t}{\psi_{loc}(h)}=\begin{cases}+1\\ -1\end{cases}$$

LOCAL
LAW OF ITERATED LOGARITHM

From (2) and (3): BM crosses level 0 infinitely many times.

PROPOSITION

The function $t\to W(t)$ from $[0,\infty)$ to $\mathbb{R}$ is CONTINUOUS but NOWHERE differentiable

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By definition: $W_{t+h}-W_t\sim\mathcal{N}(0,h)$

which implies they are symmetric around 0 with variance h, so that

$W_{t+h}-W_t\approx\pm\sqrt{h}$ with prob. $\frac{1}{2}$ each.

In very loose terms:

$$dW_t=W_t-W_{t-dt}\longrightarrow\boxed{(dW_t)^2=dt}$$

QUADRATIC VARIATION

$$[M]_t=\lim_{|\pi|\to 0}\sum_{t_i\in\pi}\left(M_{t_{i+1}}-M_t\right)^2$$

$[M]_t$ represents the realized volatility of the path

In general, $[M]_t$ is a random variable.

MARTINGALES IN CONTINUOUS TIME

L^1: $\mathbb{E}[|X|] < \infty$

LEBESGUE SPACE

A martingale w.r.t. $\mathbb{P}$ and $\mathbb{F}$ is a stochastic process $M = (M_t)$ s.t. M is adapted to $\mathbb{F}$, $M \in L^1(P)$ and satisfies:

$$\mathbb{E}[M_t | F_s] = M_s \quad \text{for } t \geq s$$

STOPPING THEOREM

M martingale with RC trajectories. σ, τ are stop. times $\sigma \leq \tau$.
If either τ is bounded by some $T \in (0, \infty)$ or M is uniformly integrable then:

- $M_\tau, M_\sigma \in L^1(P)$
- $E[M_\tau | F_\sigma] = M_\sigma$

Two frequent applications:

① M RC martingale and τ stopping time:

- $E[M_{\tau \wedge t} | F_s] = M_{\tau \wedge s}$ for $s \leq t$ so

$M^\tau = (M_t^\tau)_{t \geq 0} = (M_{t \wedge \tau})_{t \geq 0}$ is again a martingale

② M RC martingale and τ stopping time: $E[M_{\tau \wedge t}] = \mathbb{E}[M_0]$

PROPOSITION

W is a $(\mathbb{P}, \mathbb{F})$- BM. Then the following are martingale:

① W itself

② $W_t^2 - t$ for $t \geq 0$

③ $e^{\alpha W - \frac{1}{2}\alpha^2 t}$ $t \geq 0$, $\forall \alpha \in \mathbb{R}$

PROPOSITION: LAPLACE TRANSFORM

W is a BM. $a>0, b>0$. $\forall \lambda > 0$ we have.

$$\mathbb{E}\left[e^{-\lambda \tau_a}\right] = e^{-a\sqrt{2\lambda}}$$

$$\mathbb{E}\left[e^{-\lambda \sigma_{a,b}}\right] = \mathbb{E}\left[e^{-\lambda \sigma_{a,b}} \mathbb{I}_{\{\sigma_{a,b}<\infty\}}\right] = e^{-a(b+\sqrt{b^2+2\lambda})}$$

If we let $\lambda \to 0$; we get:

- $\mathbb{P}\left[\sigma_{a,b} < \infty\right] = e^{-2ab}$
- $\mathbb{P}\left[\sigma_{a,b} = +\infty\right] = 1 - e^{-2ab}$

For a general variable $U \geq 0$:

$$\boxed{\lambda \to \mathbb{E}\left[e^{-\lambda U}\right] \text{ for } \lambda > 0 \text{ is the Laplace Transform}}$$

MARKOV PROPERTIES

Suppose that a time T we want to predict the behavior of W on the basis of the past of W up to T.

Then we can forget the past and look only at the current W_T

$$\boxed{\mathbb{E}\left[g(W_u, u \geq T) \middle| \sigma(W_s, s \leq T)\right] = \mathbb{E}\left[g(W_u, u \geq T) \middle| \sigma(W_T)\right]}$$

MARKOV PROPERTY OF BM

5. STOCHASTIC INTEGRATION

The goal is to construct a stochastic integral process:

$$H \cdot M = \int H \, dM$$

- M is a local martingale null at zero (portfolio assets/ stocks)
- H is a predictable process. (portfolio weights)

In this chapter: $M=(M_t)_{t\geq 0}$ local martingale null at zero with RCLL (right continuous with left limits trajectories)

THEOREM: QUADRATIC VARIATION

For any local martingale $M=(M_t)_{t\geq 0}$ null at zero, there exists a unique RCLL process $[M]=([M]_t)_{t\geq 0}$ null at zero with $\Delta[M]=(\Delta M)^2$ and the property $M^2-[M]$ is also a local martingale.

$[M]$ is the OPTIONAL QUADRATIC VARIATION or SQUARE BRACKET PROCESS of M

COVARIATION

For two martingales M, N null at zero we define the (optional) covariation process $[M,N]$ as

$$\boxed{[M,N] := \frac{1}{4}\left([M+N]-[M-N]\right)}$$

Note that the operator $[\cdot,\cdot]$ is BILINEAR.

SHARP BRACKET (PREDICTABLE VARIANCE)

If $M \in L^2$ martingale: there exists a unique increasing predictable integrable process: $\langle M \rangle$ null at zero s.t.:

- $[M]-\langle M\rangle$ is a martingale

Therefore

- $\left(M^2-\langle M\rangle = M^2-[M]+[M]-\langle M\rangle\right)$ is a martingale

Note that, if M is real-valued, then $[M]$ becomes a $(d \times d)$ matrix-valued process with

$$[M]^{ik} = [M^i, M^k]$$

Also:

- $[M]$ exists for any local martingale null at zero
- $\langle M \rangle$ requires some extra local integrability.

BOUNDED ELEMENTARY PROCESSES

We denote by $b\mathcal{E}$ the set of all elementary processes s.t.:

$$H = \sum_{i=0}^{n-1} h_i \mathbb{I}_{(t_i, t_{i+1}]}$$

with $n \in \mathbb{N}$, $0 < t_1 \dots < t_n$ and h_i bounded $\mathcal{F}_{t_i}$-measurable random variable.

Then, for any stochastic process X:

$$\boxed{\int_0^t H_s \, dX_s := H \cdot X_t := \sum^{n-1} h_i \underbrace{\left(X_{t_{i+1} \wedge t} - X_{t_i \wedge t} \right)}_{}}$$

PRICE CHANGE

stochastic integral

min $\{t_{i+1}, t\}$

number of shares

LEMMA

M square integrable martingale.

For every $H \in b\mathcal{E}$, $H \cdot M = \int H \, dM$ is also a square-integrable martingale and we have:

- $[H \cdot M] = \int H^2 \, d[M]$
- $\mathbb{E}\left[(H \cdot M_\infty)^2 \right] = \mathbb{E}\left[\left(\int H_s \, dM_s \right)^2 \right] = \mathbb{E}\left[\int H_s^2 \, d[M_s] \right] =$

$$= \mathbb{E}\left[\sum^{n-1} h_i^2 \left([M]_{t_{i+1}} - [M]_{t_i} \right) \right]$$

isometry property

Goal: we want to expand the definition of the stochastic integral not only to $H \in b\mathcal{E}$, but to a larger class.

We define the PREDICTABLE σ-FIELD $\mathcal{P}$ on $\bar{\Omega}$ generated by all adapted left-continuous processes.

We call H PREDICTABLE if it is $\mathcal{P}$-measurable.

We define the measure $P_M = P \otimes [M]$: by

$$\int Y \, dP_M := E_M[Y] := \mathbb{E}\left[\int_0^\infty Y_{s(w)} \, d[M]_{s(w)} \right]$$

LINEARITY OF $H \cdot M$ AND M_0^2

For a fixed square-integrable M:

$\boxed{M \to H \cdot M \text{ is linear and goes from } b\mathcal{E} \text{ to } M_0^2}$
$\boxed{(M_0^2 : \text{ space of all RCLL martingales } N \text{ null at zero which } \sup_{t \geq 0} \mathbb{E}[N_t^2] < \infty)}$

$b\mathcal{E}$ represents trading strategies "a scalini" (step functions)
$L^2(M)$ represent continuous trading strategies
[ANALOGY: The closure of RATIONAL NUMBER SET $\mathbb{Q}$ is $\mathbb{R}$]
$L^2(M)$ is the closure of $b\mathcal{E}$.

PROPOSITION

Suppose M in M_0^2, then:

1. $b\mathcal{E}$ is dense in $L^2(M)$, the closure of $L^2(M)$ is $L^2(M)$
   (Every $H \in L^2(M)$ can be written as a limit of a sequence $(H^n)_{n \in \mathbb{N}}$ in $b\mathcal{E}$)
2. For every $H \in L^2(M)$, $H \cdot M = \int H \, dM$ is well defined in M_0^2 and satisfies isometry property.

⚠ Note that Brownian motion has $\mathbb{E}[W_t^2] = t$ so $\sup_{t \geq 0} \mathbb{E}[W_t^2] = +\infty$, so $M \notin M_0^2$.

(Reason why we localize M with a stopping sequence)

LOCALLY SQUARE INTEGRABLE MARTINGALE

M locally square integrable martingale: $M \in \mathcal{M}^2_{0,loc}$
if there exists a sequence of stopping times $\tau_n \to \infty$ P-a.s.
s.t. $M^{\tau_n} \in \mathcal{M}^2_0$ $\forall n$.

H predictable process locally in $L^2(M)$: $H \in L^2_{loc}(M)$ if there exists a sequence of stopping times τ_n s.t. $H^{\tau_n} \in L^2(M)$ $\forall n$.

For $M \in \mathcal{M}^2_{0,loc}$, $H \in L^2_{loc}(M)$:

$$H \cdot M := \left(H \, \mathbb{I}_{[0,\tau_n]}\right) \cdot M^{\tau_n}$$

Another definition of $L^2_{0,loc}$:

$$L^2_{0,loc}(M) = \left\{ \text{all predictable processes } H = (H_t)_{t \geq 0} \text{ s.t. } \int_0^t H_s^2 \, d[M]_s < \infty \text{ P-a.s. } \forall t \geq 0 \right\}$$

PROPERTIES

LOCAL MARTINGALES PROPERTIES

- If H local martingale, $H \in L^2_{loc}(M)$: $\int H dM \in M^2_{\emptyset,loc}$ local martingale.
  If $H \in L^2(M)$: $\int H dM \in M^2_\emptyset$
- If M loc. mart. and H predict. locally bounded) $\to$ $\int H dM$ is a loc. mart.
  example: continuous adapted process
- If $M \in M^2_\emptyset$ mart. and H predict. and bounded: $\to$ $\int H dM \in M^2_\emptyset$ mart.

LINEARITY

- If M loc. mart, $H, H' \in L^2_{loc}(M)$, $a, b \in \mathbb{R}$: $aH + bH' \in L^2_{loc}(M)$ and:

$$(aH + bH') \cdot M = (aH) \cdot M + (bH') \cdot M = a(H \cdot M) + b(H' \cdot M)$$

ASSOCIATIVITY

- Predictable process $K \in L^2_{loc}(H \cdot M)$ iff $K \cdot H \in L^2_{loc}(M)$, Then:

$$K \cdot (H \cdot M) = (KH) \cdot M$$

BEHAVIOR UNDER STOPPING

- $(H \cdot M)^\tau = H \cdot (M^\tau)$

(we can either stopping and then integrating or the other way around)

QUADRATIC VARIATION

$H, K \in L^2_{loc}(M)$, M, N local martingales

- $\left[\int H \, dM, N\right] = \int H \, d[M, N]$
- $\left[\int H dM, \int K dN\right] = \int (HK) \, d[M, N]$

JUMPS

- $\Delta\left(\int H\,dM\right)_t = H_t\,\Delta M_t$

where $\Delta M_t = M_t - M_{t-}$

EXTENSIONS TO SEMIMARTINGALES

SEMIMARTINGALES

A semimartingale is a stochastic process $X = (X_t)_{t\geq 0}$ that can be decomposed as:

$$X = X_0 + M + A$$

M: local martingale null at zero

A: adapted RCLL process null at zero

If A is PREDICTABLE: X is a special martingale.

Note that the decomposition of X is not UNIQUE in general, but:

- If X special martingale: the decomposition is unique and called CANONICAL
- If X continuous: M and A continuous and therefore X special.

OPTIONAL QUADRATIC VARIATION OF SEMIMART.

X semimartingale:

$$[X] = [M] + 2[M,A] + [A] = [M] + 2\sum \Delta M \Delta A + \sum (\Delta A)^2$$

or

$$[X] = X^2 - 2\int X_-\,dX$$

EXAMPLE

$$X_t = X_0 + \int_0^t \alpha_s \, dW_s + \int_0^t \beta_s \, ds$$

SEMIMARTINGALE

W BROWNIAN MOTION

α, β predictable s.t.

$$\int_0^t \left(\alpha_s^2 + |\beta_s|\right) ds < +\infty$$

STOCHASTIC INTEGRAL WITH SEMIMART.

X semimartingale

$$H \cdot X = H \cdot M + H \cdot A$$

with the following properties:

- $H \cdot X$ semimartingale
- If X special with canonical decomposition $X = X_0 + M + A$, then $H \cdot X$ special with $H \cdot X = H \cdot M + H \cdot A$

(+ all the previous properties)

6. STOCHASTIC CALCULUS

If X is a semimartingale, f a suitable function: what can we say about $f(X)$?!

RECALL: for a semimartingale $X = X_0 + M + A$

- $$[A]_t = \sum_{0 \le s \le t} (\Delta A_s)^2 = \sum_{s \le t} (A_s - A_{s-})^2$$
$$= \lim_{n \to \infty} \sum_{t_i \in \Pi_n} (A_{t_{i+1} \wedge t} - A_{t_i \wedge t})^2$$
- $$[A, Y]_t = \sum_{0 \le s \le t} \Delta A_s \Delta Y_s = \lim_{n \to \infty} \sum (A_{t_{i+1} \wedge t} - A_{t_i \wedge t})(Y_{t_{i+1} \wedge t} - Y_{t_i \wedge t})$$
  for semimartingale Y.

So the quadratic variation of a general semimartingale $X = X_0 + M + A$

$$[X] = [M + A] = [M] + [A] + 2[M, A] =$$
$$= [M] + \sum_{s \le t} (\Delta A_s)^2 + 2 \sum_{s \le t} \Delta M_s \Delta A_s$$

If semimartingale X is continuous: $[A] = \emptyset$ and then:

- $[X] = [M]$
- $[M] = \langle M \rangle$

$$\boxed{[X] = [M] = \langle M \rangle}$$

- $$\boxed{\lim_{n \to \infty} \sum_{\Pi_n} (M_{t_{i+1} \wedge t} - M_{t_i \wedge t})(A_{t_{i+1} \wedge t} - A_{t_i \wedge t}) = \emptyset}$$

CAUCHY SCHWARTZ

ITO'S FORMULA (1)

X continuous martingale, $f: \mathbb{R} \to \mathbb{R}$ in C^2

$f(X) = \left(f(X_t) \right)_{t \geq \emptyset}$ is a CONTINUOUS MARTINGALE

$$\boxed{f(X_t) = f(X_0) + \int_\emptyset^t f'(X_s)\,dX_s + \frac{1}{2}\int_\emptyset^t f''(X_s)\,d\langle X\rangle_s}$$

differential notation

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle X\rangle_t$$

$$= f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle M\rangle_t$$

$Y = f(X)$ price of a financial derivative with underlying asset X.
How does Y change based on X?

[The linear approximation is NOT enough, we must account for the second-order behavior of X]

ITO'S FORMULA (2)

X general $\mathbb{R}^d$-valued semimartingale, $f: \mathbb{R} \to \mathbb{R}$ in C^2
$f(X)$ is a martingale and:

1. if X has continuous trajectories:

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_\emptyset^t \frac{\delta f}{\delta X_i}(X_s)\,dX_s^i + \frac{1}{2}\sum_{i=1}^{d} \int_\emptyset^t \frac{\delta^2 f}{\delta x^i \delta x^j}(X_s)\,d\langle X^i, X^j\rangle$$

2. if $d = 1$ (equivalent to X real-valued BUT NOT NECESSARILY CONTINUOUS)

$$f(X_t) = f(X_0) + \int_\emptyset^t f'(X_{s-})\,dX_s + \frac{1}{2}\int_\emptyset^t f''(X_{s-})\,d[X_s]$$

$$+ \sum \left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_s)(\Delta X_s)^2 \right)$$

Note: if X^K has finite variation: we have a nice simplification

$$\langle X^K \rangle = \emptyset$$

$$\langle X^K, X^l \rangle \emptyset$$

EXAMPLE: CRR BINOMIAL MODEL

In discrete Time:

$$\frac{\tilde{S}^0_K - \tilde{S}^0_{K-1}}{\tilde{S}^0_{K-1}} = r$$

$$Y_K = \frac{\tilde{S}^1_K}{\tilde{S}^1_{K-1}}$$

$$\frac{\tilde{S}^1_K - \tilde{S}^1_{K-1}}{\tilde{S}^0_{K-1}} = \underbrace{Y_K - 1}_{R_K \text{ NET RETURN}} = R_K = \mathbb{E}[R_K] + (R_K - \mathbb{E}[R_K])$$

From steps of size 1 to dt:

$$\boxed{\begin{gathered}\frac{d\tilde{S}^0_t}{\tilde{S}^0_t} = r\,dt \\ \frac{d\tilde{S}^1_t}{S^1_t} = \mu\,dt + \sigma\,dW_t\end{gathered}}$$

STOCHASTIC DIFFERENTIAL EQUATION OF STOCK PRICE

The solution for this SDE is given by

$$\boxed{\tilde{S}^1_t = \tilde{S}^t_0 \exp\Big(\sigma W_t + \underbrace{(\mu - \tfrac{1}{2}\sigma^2)t}_{}\Big)} \qquad t \geq 0$$

extra term $-\frac{1}{2}\sigma^2 t$ given by ITO's

IN GENERAL ✳

X continuous semimartingale null at zero

$$dZ_t = Z_t\,dX_t \qquad Z_0 = 1$$

The unique solution is:

$$\boxed{Z_t = e^{X_t - \frac{1}{2}\langle X \rangle_t}}$$

put differently, process Z satisfies:

$$Z_t = 1 + \int_0^t Z_s\,dX_s$$

STOCHASTIC EXPONENTIAL

X semimartingale null at zero

The STOCHASTIC EXPONENTIAL of X is the unique solution Z_t to:

$$\boxed{dZ_t = Z_{t-}\, dX_t, \qquad Z_0 = 1}$$

(The only semimartingale satisfying: $Z_t = 1 + \int_0^t Z_{s-}\, dX_s$

and it is denoted $\boxed{\mathcal{E}(X) = Z}$

For example * where Z is continuous we have the explicit formula

$$\mathcal{E}(X) = \exp\left(X - \frac{1}{2}\langle X \rangle\right)$$

ITO'S PROCESS

$$X_t = X_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s \qquad \text{for } t \geq 0$$

for some $(\mathbb{P}, \mathbb{F})$-Brownian motion W

μ, σ predictable process with integrability conditions $\left[\int_0^T |\mu_s| + |\sigma_s|^2 ds < \infty\right]$

[X, μ, W_s can be vector, σ matrix]

for any $f \in C^2$: also $f(x)$ is a ITO'S PROCESS

$$f(X_t) = f(X_0) + \int_0^t \left(f'(X_s)\mu_s + \frac{1}{2} f''(X_s)\sigma_s^2\right) ds + \int_0^t f'(X_s)\sigma_s\, dW_s$$

PRODUCT RULE

X, Y ACLL semimartingales

$$\boxed{X_t Y_t = \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X,Y]_t}$$

$$\left[d(XY) = X_- dY + Y_- dX + d[X,Y] \right)$$ differential notation

If X, Y continuous:

$$d(X,Y) = X_- dY + Y_- dX + d\langle X, Y\rangle$$

SOME USEFUL RESULTS FOR BM AND $[a,b]$

W BROWNIAN MOTION, $a < 0 < b$

$$\tau_{a,b} = \inf\{t \geq 0 : W_t > b \text{ OR } W_t < a\}$$ first time W leaves $[a,b]$

$$\boxed{\begin{aligned} &\mathbb{E}[\tau_{a,b}] = |a| b \\ &P[W_{\tau_{ab}} = b] = \frac{|a|}{b-a} = 1 - P[W_{\tau_{a,b}} = a] \\ &\mathrm{Cov}(\tau_{a,b}, W_{\tau_{ab}}) = \mathbb{E}[\tau_{a,b} W_{\tau_{ab}}] = \frac{1}{3}\mathbb{E}[W^3_{\tau_{a,b}}] = \frac{1}{3}|a| b(b-|a|) \end{aligned}}$$

GIRSANOV'S THEOREM

The goal is to show that the class of semimartingales is also invariant under a change to an equivalent probability measure.

$$\mathbb{Q} \approx \mathbb{P} \quad \text{on } \mathcal{F}_T, \; T \in (0, \infty)$$

$$\mathbb{Q} \overset{loc}{\approx} \mathbb{P} \quad \text{on } \mathcal{F}_T, \; T < \infty$$

$$Z_t = Z_t^{Q,P} = E_P\left[\frac{dQ|_{F_T}}{dP|_{F_T}}\middle| F_T\right]$$

density of $\mathbb{Q}$ with respect to $\mathbb{P}$

LEMMA

$\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ with $Z = Z^{Q,P}$

1. For $s \le t \le T$, U_t $\mathcal{F}_t$-measurable (either ≥ 0 or in $L^1(\mathbb{Q})$)

$$\boxed{E_{\mathbb{Q}}\left[U_t \mid \mathcal{F}_s\right] = \frac{1}{Z_s} E_P\left[Z_t U_t \mid \mathcal{F}_s\right]}$$

BAYES FORMULA

2. Y (local) martingale on $\mathbb{Q}$ $\longleftrightarrow$ ZY (local) martingale on $\mathbb{P}$ in $[0,T]$

GIRSANOV'S THEOREM

$\mathbb{Q} \overset{loc}{=} \mathbb{P}$, $Z = Z^{Q,P}$, M local $\mathbb{P}$-martingale null at 0:

$$\tilde{M} := M - \int \frac{1}{Z} d[Z, M] \quad \text{local } \mathbb{Q}\text{-martingale}$$

(every $\mathbb{P}$-martingale is a $\mathbb{Q}$-martingale and v.v.)

STOCHASTIC LOGARITHM

Given a positive process Z, its STOCHASTIC LOGARITHM L is:

$$\boxed{dL_t = \frac{dZ_t}{Z_{t-}} \quad \text{or} \quad L_t = \int_0^t \frac{1}{Z_{s-}} dS_-}$$

With the property: $Z_t = \mathcal{E}(L)_t$ STOCHASTIC EXPONENTIAL

(analagous of $d(\ln(Z_t)) = \frac{dZ_t}{Z_t}$ in standard calculus $Z(t)$)

GIRSANOV'S THEOREM (CONTINUOUS)

$Q \overset{loc}{\approx} P$, $Z^{Q,Z}$ continuous, $Z = Z_0 \mathcal{E}(L)$

M P-martingale null at 0.

$$\tilde{M} = M - [L, M] = M - \langle L, M \rangle \quad Q\text{-martingale}$$

Specifically, if W is a P-Brownian Motion, $\tilde{W}$ is a Q-Brownian Motion

In particular, if $L = \int v \, dW$:

$$\boxed{\tilde{W} = W - \langle \int v \, dW, W \rangle = W - \int v_s \, ds}$$

we substract a drift to obtain under Q a BM

Consider now F^W Filtration generated by W Brownian Motion

ITO'S REPRESENTATION PROBLEM

W Brownian motion, every random variable $H \in L^1(\mathbb{F}^W_\infty, \mathbb{P})$ has a unique representation as:

$$\boxed{H = \mathbb{E}[H] + \int_0^\infty \psi_s \, dW_s}$$

ITO'S DECOMPOSITION

INTUITION:

- H: payoff
- $\mathbb{E}[H]$ initial capital (fair price of the option today) $(\mathbb{E}[H] = H_0)$
- ψ_s hedging strategy
- $\int \psi \, dW$ the Trading gains

LEMMA

1. Every local $(\mathbb{P}, \mathbb{F}^W)$-martingale is of the form $L = L_0 + \int v \, dW$
2. Every $(\mathbb{P}\text{-}\mathbb{F}^W)$-martingale is CONTINUOUS

No Jumps!

DUDLEY THEOREM

W Brownian motion, then any random variable H can be written as:

$$H = \int_0^\infty \psi_s \, dW_s$$

(INTUITION: we can replicate any payoff with a trading strategy.
IT looks like ARBITRAGE, but IT IS NOT a valid strategy because
ψ_s is NOT ADMISSIBLE)

In general: $\int \psi \, dW$ is a LOCAL MARTINGALE, not a TRUE M.

7. BLACK-SCHOLES

$(\Omega, \mathcal{F}, \mathbb{P})$, W Brownian Motion

$\mathbb{F} = \mathcal{F}_t$ filtration generated by W

The financial market has 2 basic traded assets:

$$\boxed{\begin{aligned} \tilde{S}^0_t &= e^{rt} && \text{BANK ACCOUNT} \\ \tilde{S}^1_t &= S^1_0 \exp\left(\sigma W_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right) && \text{RISKY ASSET (STOCK)} \end{aligned}}$$

Applying Ito's formula:

$$d\tilde{S}^0_t = \tilde{S}^0_r r\,dt \qquad \frac{d\tilde{S}^0_t}{\tilde{S}^0_t} = r\,dt$$

GROWTH RATE OF BANK ACCOUNT

$$d\tilde{S}^1_t = \tilde{S}^1_t(\mu\,dt + \sigma\,dW_t) \qquad \frac{d\tilde{S}^1_t}{\tilde{S}^1_t} = \mu\,dt + \sigma^2 dW_t$$

DRIFT

MEAN 0
VARIANCE σ^2
(INSTANEOUS VOLATILITY σ)

If we consider DISCOUNTED value:

$$S^0 = 1$$

$$S^1_t = S^t_0 \exp\left(\sigma W_t + \left(\mu - r - \tfrac{1}{2}\sigma^2\right)t\right)$$

We want to find an EQUIVALENT MARTINGALE MEASURE for S^1

$$dS^1_t = S^1_t \sigma\left(dW_t + \underbrace{\frac{r-\mu}{\sigma}}_{\lambda} dt\right) = S^1_t \sigma\, dW^*_t$$

$$\boxed{W^*_t = W_t + \frac{\mu - r}{\sigma} t = W_t + \int_0^t \lambda\, ds}$$

INTUITION
λ represents how much we should shift $\mathbb{P}$ to obtain $\mathbb{Q}$ s.t. is a "fair" game

$$\boxed{\lambda = \frac{\mu - r}{\sigma}}$$

MARKET PRICE OF RISK //
INFINITESIMAL SHARPE RATIO

W^* is a Brownian Motion under $\mathbb{Q}^* \approx \mathbb{P}$ given by:

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} := \mathcal{E}\left(-\int \lambda \, dW\right)_T = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right)$$

$$\boxed{Z^{\mathbb{Q}^*,\mathbb{P}} = Z^*_t = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right)}$$

so we get

$$\boxed{S^1_t = S^1_0 + \int_0^t S^1_u \sigma \, dW^*_u}$$

CONTINUOUS $\mathbb{Q}$-MARTINGALE | RISK-NEUTRAL DYNAMICS

⚠ $\mathbb{Q}^*$ is UNIQUE!

existence of $\mathbb{Q}^* \longleftrightarrow$ NA

uniqueness of $\mathbb{Q}^* \longleftrightarrow$ completeness

---

ITO'S REPRESENTATION OF PAYOFF

(MARTINGALE APPROACH TO VALUING OPTIONS)

H random discounted payoff

If $H \in L^1(\mathbb{Q}^*)$:

$$V^*_t := \mathbb{E}_{\mathbb{Q}^*}[H \mid \mathcal{F}_T] = \mathbb{E}[H] + \int_0^t \varphi^H_s \, dW^*_s$$

$\mathbb{Q}^*$-martingale

Ito's Representation Theorem

we can now define:

$$\theta^H_t = \frac{\varphi^H_T}{S^1_t \sigma}$$

$$\eta^H_t = V^*_t - \theta^H_t S^1_t$$

$\varphi^H(\theta^H, \eta^H)$ s.t.

REPLICATING STRATEGY FOR H

$$\boxed{V_t(\varphi^H) = \theta^H_T S^1_T + \eta^H_t S^0_t = V^*_t = H}$$

(Note that Ito's representation gives the existence of the strategy but it doesn't give practical tools).

MARKOVIAN PAYOFFs AND PDEs

Goal: for a general payoff H, compute the discounted value process V^* and the strategy ϑ^H explicitly.

We start with the value process V_t^*

$$V_t^* = E_{Q^*}[H \mid \mathcal{F}_t] = E_{Q^*}\left[h(S_T^1) \mid \mathcal{F}_t\right]$$

(discounted arbitrage-free price of the derivative H)

$$S_T^1 = S_t^1 \frac{S_T^1}{S_t^1} = \underbrace{S_t^1}_{\mathcal{F}_t\text{- measurable}} \exp\Big(\sigma \underbrace{(W_T^* - W_t^*)}_{N(0,T-t)} - \frac{1}{2}\sigma^2 (T-t)\Big)$$

$$\boxed{V_t^* = E_{Q^*}\left[h(S_T^1) \mid \mathcal{F}_t\right] = v(t, S_t^1)}$$

$$v(t,x) = \int_{-\infty}^{\infty} h\left(x e^{\sigma\sqrt{T-t}\, y - \frac{1}{2}\sigma^2 (T-t)}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy$$

$$dV_t^* = dv(t, S_t^1)$$

$$dV_t^* = v_x(t, S_t^1)\sigma S_t^1 \, dW_t^* + \left(v_t(t, S_t^1) + \frac{1}{2} v_{xx}(t, S_t^1)\sigma^2 (S_t^1)^2\right) dt$$

$$v_x(t, S_t^1)\, dS_t^1 = dV_t^* = \vartheta_t^H \, dS_t^1$$

$$\boxed{\vartheta_t^H = \frac{\partial v}{\partial x}(t, S_t^1)}$$

STRATEGY

$x : S_t^1$

we also note that $v(t,x)$ must satisfy:

$$0 = \frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}$$

and the boundary condition:

$$v(T, \cdot) = h(\cdot)$$

THE BLACK-SCHOLES FORMULA

For a european call option, the value process can be computed explicitly.

$$\tilde{H} = (\tilde{S}_T^1 - \tilde{K})^+ \quad \text{undiscounted payoff}$$

$$H = \frac{\tilde{H}}{\tilde{S}_T^0} = (S_T^1 - \tilde{K}e^{-rT})^+ = (S_T^1 - K)^+$$

we obtain that the discounted value of H is:

$$V_t^H = V_t^* = \mathbb{E}_{Q^*}[H \mid \mathcal{F}_t] =$$

$$\boxed{V_t^H = V_t^* = \mathbb{E}_{Q^*}\left[\left(x e^{\sigma\sqrt{T-t}\,Y - \frac{1}{2}\sigma^2(T-t)} - K\right)^+\right]_{x = S_t^1} \quad \text{with } Y \sim N(0,1)}$$

An elementary computation with normal distr. leads to

$$\mathbb{E}_{Q^*}\left[\left(x e^{aY - \frac{1}{2}a^2} - b\right)^+\right] = x\,\Phi\left(\frac{\log(\frac{x}{b}) + \frac{1}{2}a^2}{a}\right) + b\,\Phi\left(\frac{\log\frac{x}{b} - \frac{1}{2}a^2}{a}\right)$$

$\Phi(y) = Q[Y \leq y]$

cumulative distribution of $N(0,1)$

If we insert

$x = S_T^1$

$a = \sigma\sqrt{T-t}$

$b = K$

and then we pass to $S_t^1 = \tilde{S}_t^1 e^{-rt}$ $\quad K = \tilde{K}e^{-rT}$

$$\boxed{\tilde{V}_t^{\tilde{H}} = \tilde{V}(t, S_t^1) = \tilde{S}_t^1\,\Phi(d_1) - Ke^{-r(T-t)}\,\Phi(d_2)}$$

BLACK SCHOLES (UNDISCOUNTED VALUES)

$$\boxed{d_{1,2} = \frac{\log\left(\tilde{S}^1_t / \tilde{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}}$$

! note that the drift μ does NOT appear

To compute the replicating strategy:

$$\theta^H_t = \frac{\delta v}{\delta x}(t, S^1_t) = \frac{\delta \tilde{v}}{\delta \tilde{x}}(t, \tilde{x})$$

In the case of Black-Scholes:

$$\theta^H_t = \Phi\left(\frac{\log\left(\tilde{S}^1_t / \tilde{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

which lies always in $[0, 1]$

GREEKS

- DELTA $= \dfrac{\delta V_t}{\delta S^1_t}$ (HEDGE RATIO)
- GAMMA $= \dfrac{\delta^2 V_t}{(\delta S^1_t)^2}$
- RHO $= \dfrac{\delta V_t}{\delta r}$
- VEGA $= \dfrac{\delta V_t}{\delta \sigma}$
- THETA $= \dfrac{\delta V_t}{\delta (T-t)}$
- VANNA $= \dfrac{\delta \text{ DELTA}}{\delta \sigma} = \dfrac{\delta^2 V_t}{\delta S^1_t \, \delta \sigma}$
- VOMMA $= \dfrac{\delta^2 V_t}{\delta \sigma^2}$
- CHARM $= \dfrac{\delta \text{ DELTA}}{\delta (T-t)}$