MATHEMATICAL FOUNDATION FOR FINANCE - CHEATSHEET

σ - ALGEBRA

$\mathcal{F}$ is a σ-algebra if

- $\emptyset, \Omega \in \mathcal{F}$
- $A \in \mathcal{F} \Leftrightarrow A^c \in \mathcal{F}$
- $(A_n)_{n \in \mathbb{N}}$ $A_n \in \mathcal{F} \Leftrightarrow (\cup A_n) \in \mathcal{F}$

CONDITIONAL EXPECTATION

$$\left.\begin{array}{l} \bullet\ Z \quad \mathcal{G}\text{-measurable and integrable} \\ \bullet\ \mathbb{E}[X 1_A] = \mathbb{E}[Z 1_A] \quad \text{for all } A \in \mathcal{G} \end{array}\right\} \begin{array}{l} Z \text{ CONDITIONAL EXPECTATION} \\ \text{OF } X \text{ GIVEN } \mathcal{G} \end{array}$$

JENSEN INEQUALITY

$$\boxed{f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]}$$

If $f = aX + b$ AFFINE FUNCTION

$\mathbb{E}[f(x)] = f(\mathbb{E}[x])$

COST PROCESS

$$\Delta C_k = (\varphi_k^0 - \varphi_{k-1}) S_k^0 + (\vartheta_k - \vartheta_{k-1}) S_k^1$$

$$C_k = C_0 + \sum_{j=1}^{k} \Delta C_k$$

SELF FINANCING:

$C_k = C_0 = V_0 = \varphi^0$

$$\boxed{\begin{aligned} V_k &= V_0 + G_k \\ &= V_0 + \sum_{j=1}^{k} \vartheta_j \Delta S_j \\ &= V_0 + G_k \end{aligned}}$$

STOPPING TIME

τ STOPPING TIME: $\{\tau \leq k\} \in \mathcal{F}_k$

TRADING TRATEGY

$\varphi = (\varphi^0, \vartheta)$

- φ_k^0 REAL-VALUED and ADAPTED
- ϑ_k $\mathbb{R}^D$-VALUED and PREDICTABLE, $\vartheta_k^0 = 0$

FATAU'S LEMMA

$(X_n)_{n \in \mathbb{N}}$ non negative random var

$$\mathbb{E}\left[\lim_{n \to \infty} [\inf(X_n)]\right] \leq \lim_{n \to \infty} \left(\inf \mathbb{E}[X_n]\right)$$

MONOTONE CONVERGENCE THEOREM

$(X_n)_{n\in\mathbb{N}}$ s.t. $X_0 \leq X_1 \leq \dots \leq X_n$, $X_n \to X$

$$\mathbb{E}\left[\lim_{n\to\infty}(X_n)\right] = \lim_{n\to\infty}\left(\mathbb{E}[X_n]\right)$$

DOMINATED CONVERGENCE THEOREM

$(X_n)_{n\in\mathbb{N}}$ integrable

$$\mathbb{E}[X] = \lim_{n\to\infty}\left(\mathbb{E}[X_n]\right) = \mathbb{E}\left[\lim_{n\to\infty}(X_n)\right]$$

ARBITRAGE OPPORTUNITY

Self-financing strategy with ZERO INITIAL WEALTH s.t.

- $V_T(\varphi) \geq 0$
- $\mathbb{P}[V_T(\varphi) > 0] > 0$

- FIRST KIND: $\mathbb{P}[V_T(\varphi) > 0] > 0$
- SECOND KIND: $\mathbb{P}[V_0(\varphi) < 0] > 0$

DOOB'S DECOMPOSITION

$X = (X_k)_{k\in\mathbb{N}_0}$ integrable and predictable

$X = X_0 + M + A$

- $\Delta A_k = \mathbb{E}[\Delta X_k \mid \mathcal{F}_{k-1}] \longrightarrow A = \sum \mathbb{E}[\Delta X_1] = k\,\mathbb{E}[\Delta X_1]$
- $\Delta M_k = \Delta X_k - \mathbb{E}[\Delta X_k \mid \mathcal{F}_{k-1} \longrightarrow M_k = \sum(\Delta X_j - \mathbb{E}[\Delta X_1)$

$\mathbb{Q} \approx \mathbb{P}$, DENSITY PROCESSES

$$\left.\begin{aligned} \mathbb{Q}[A] &= \mathbb{E}_{\mathbb{P}}[Z_k \mathbb{1}_A] \\ \mathbb{E}_{\mathbb{Q}}[Y] &= \mathbb{E}_{\mathbb{P}}[Z_k Y] \end{aligned}\right\} \quad Z_k = \frac{d\mathbb{Q}}{d\mathbb{P}}$$

BINOMIAL MODEL $\mathbb{Q} \approx \mathbb{P}$

$\mathbb{Q} \approx \mathbb{P}$ s.t. $\tilde{S}^1/\tilde{S}^0$ $\mathbb{Q}$-martingale iff $u > r > d$.

$$\mathbb{Q}[Y_k = 1+u] = q^* = \frac{r-d}{u-d} = 1 - \mathbb{Q}[Y_k = 1+d]$$

FOUNDAMENTAL THEOREM OF ASSET PRICING

S with $S^0 \equiv 1$ is ARBITRAGE FREE if

$$\boxed{\text{NA for } S \longleftrightarrow \mathbb{P}_{e,loc}(S) \neq \emptyset \longleftrightarrow \mathbb{P}_e(S) \neq \emptyset}$$

ATTAINABLE PAYOFFS $\mathcal{H}$

$\tilde{H}$ attainable if $V_K(\varphi) = \dfrac{\tilde{H}}{1+r}$

(φ admissible - self-financing)

$$V_0 + \sum_{j=1}^{K} \theta_j \Delta S_j^1 = \frac{\tilde{H}}{(1+r)^K}$$

BROWNIAN MOTION

STOCHASTIC PROCESS $W = (W_t)_{t \geq 0}$

(BM1) $W_0 = 0$

(BM2) $W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$

(BM3) W has continuous trajectories

$\boxed{(dW_t)^2 = dt}$

$\boxed{E[W_t] = 0}$

$\boxed{[W_t] = t}$

PROPERTIES OF BM

- $W^1 := -W$ is a BM
- $W^2_t := W_{t+T} - W_T$ is a BM, $t, T > 0$
- $W^3_t := c W_{\frac{t}{c^2}}$ is a BM, $c \neq 0$ (RESCALING PROPERTY)
- $W^4_t := W_{T-t} - W_T$ is a BM, on $[0,T]$ $t \leq T$
- $W^5_t = \begin{cases} t W_{\frac{1}{t}} & t > 0 \\ 0 & \text{else} \end{cases}$ is a BM

⚠
- W^2
- e^{aW}

SUB MARTINGALES (NOT MARTINGALES!)

GEOMETRIC BROWNIAN MOTION

$$\boxed{S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}}$$

- $\lim_{t \to \infty} S_t = \begin{cases} +\infty & m > 0 \\ 0 & m < 0 \end{cases}$
- $E[S^t] = S_0 e^{\mu t}$
- S^t SUB/SUPER/MARTINGALE $\Longleftrightarrow$ $m + \sigma^2 \geq / \leq / = 0$

In general: if $X \sim N(\mu, \sigma^2)$, $a > 0$

$$\mathbb{E}(e^{aX}) = e^{\frac{1}{2}a^2\sigma^2 + a\mu}$$

QUADRATIC VARIATION

Any martingale M null at 0 has unique RCLL $[M]$ s.t.

$\Delta[M] = (\Delta M)^2$ and $M^2 - [M]$ LOCAL MARTINGALE

$$\boxed{[M]_t = \lim_{\substack{\Pi \to 0 \\ t_i \in \Pi}} (M_{t_{i+1}} - M_{t_i})^2}$$

- $[M+N] = [M] + [N] + 2[X,N]$ COVARIATION 0 IF INDEPENDENT

• COVARIATION IS BILINEAR
• $[M,M] = [M]$

- $[M,N] = \frac{1}{4}\left([M+N] - [M-N]\right)$ COVARIATION M,N local martingales
- $\boxed{\left[\int H\,dW\right] = \int H^2\,d[W]}$
- $\left[\int H\,dM,N\right] = \int H\,d[M,N]$
- $\left[\int H\,dM, \int K\,dN\right] = \int (HK)\,d[M,N]$

SHARP BRACKETS (SHARP VARIANCE)

unique increasing integrable process $\langle M \rangle$ s.t.

$[M] - \langle M \rangle$ is a MARTINGALE

($M^2 - \langle M \rangle = M^2 - [M] + [M] - \langle M \rangle$ is also a MARTINGALE)

IF M CONTINUOUS: $\langle M \rangle$ PREDICTABLE

SETS

- $\mathbb{P}_e(S)$: equivalent martingale measures (EMMs) for S
- $\mathbb{P}_{e,loc}(S)$: equivalent local martingale measures (ELMMs) for S
- M_0^2 : square-integrable martingales null at 0 $\left(\begin{array}{l} \sup \mathbb{E}[W_t^2] = t \\ W \notin M_0^2 \end{array}\right)$
- $M_{0,loc}^2$ square-integrable local martingales null at 0
- $b\mathcal{E}$: bounded elementary predictable processes $H_t = \sum H_i \mathbb{1}_{(T_i, T_{i+1})}$
- $L^2(M)$: predictable processes square-integrable w.r.t. M

ISOMETRY PROPERTY

$$\mathbb{E}\left[(H\cdot M_\infty)^2\right] = \mathbb{E}\left[\left(\int_0^\infty H\, dM_s\right)^2\right] = \mathbb{E}\left[\int_0^\infty H_s^2\, d[M]_s\right]$$

ITO'S FORMULA

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s$$

$$f(X_t) = f(X_0) + \int_0^t \frac{\delta f}{\delta t}\,ds + \int_0^t \frac{\delta f}{\delta X}\,dX + \frac{1}{2}\int_0^t \frac{\delta f}{\delta X^2}\,d\langle X\rangle_s$$

$$f(X_t, Y_t) = f(X_0, Y_0) + \int_0^t \frac{\delta f}{\delta X}\,dX_s + \int_0^t \frac{\delta f}{\delta y}\,dY_s + \frac{1}{2}\int_0^t \frac{\delta^2 f}{\delta x^2}\,d[X]_s + \frac{1}{2}\int_0^t \frac{\delta^2 f}{\delta y^2}\,d[Y_s] + \int_0^t \frac{\delta f}{\delta x\,\delta y}\,d[X_s, Y_s]$$

X not necessarily continuous

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})\,dX_s + \frac{1}{2}\int_0^t f''(X_{s-})\,d[X]_s$$
$$+ \sum_{0\le s\le t}\left(f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s - \frac{1}{2} f''(X_{s-})(\Delta X_s)^2\right)$$

ITO'S FORMULA: USEFUL RESULT

$(f(t, W_t))_{t\geq 0}$ is a CONTINUOUS LOCAL MARTINGALE IFF

$$\int_0^t \left[\frac{\delta f}{\delta t}(s, W_s) + \frac{1}{2}\frac{\delta^2 f}{\delta t^2}(s, W_s)\right] ds = 0$$

PRODUCT RULE

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_{s-}\,dX_s + \int_0^t X_{s-}\,dY_s + [X, Y]_t$$

$$d(XY) = Y\,dX + X\,dY + d\langle X, Y\rangle$$

STOCHASTIC EXPONENTIAL

$$\boxed{Z_t = \mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)}$$

Z_t unique solution to: $Z_t = 1 + \int Z_s \, dX_s$

$$d\mathcal{E}(X)_t = \mathcal{E}(X)_t \, dX_t$$

GIRSANOV THEOREM

$\mathbb{Q} \overset{LOC}{\approx} \mathbb{P}$ with density process

M $\mathbb{P}$-martingale, then $\tilde{M}$ $\mathbb{Q}$-martingale

$$\tilde{M} = M - \int \frac{1}{Z} \, d[Z, M]$$

$Z = Z_0 \, \mathcal{E}(L)$

$$\tilde{M} = M - [L, M]$$

Z continuous

$L = \int v \, dW$

$$\tilde{W} = W - \int v_s \, ds$$

W Brownian motion

DENSITY PROCESS OF $\mathbb{Q}$ W.R.T. $\mathbb{P}$

$$Z_t^{\mathbb{Q},\mathbb{P}} = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_{\mathcal{F}_T}}{d\mathbb{P}_{\mathcal{F}_T}} \,\middle|\, \mathcal{F}_t \right]$$

$Z > 0$ on $[0,T]$

Z (super)martingale

ITO'S REPRESENTATION

$$H = \mathbb{E}[H] + \int_0^T \psi_s \, dW_s$$

H martingale

W Brownian motion