FINANCIAL RISK MANAGEMENT

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NOTES BY GIACOMO MAGGIORE, FALL 2025

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VECTORS AND MATRICES

BASIC INTRO OF VECTORS, MATRICES $\longrightarrow$ SKIPPED

SYSTEMS OF EQUATIONS AND QUADRATIC FORMS

$$Ax = b$$

$A_{n \times m}$ coefficient matrix

$x = (x_1 \dots x_m)^T$ $m \times 1$ vector of unknowns

$b = (b_1 \dots b_n)^T$ $n \times 1$ vector of known values

If $n = m$ and A^{-1} exists:

$x = A^{-1} b$ (EXCEL: MINVERSE)

QUADRATIC FORM

A SQUARE MATRIX $n \times n$

x $n \times 1$ VECTOR

$x^T A x$ IS CALLED QUADRATIC FORM

EXAMPLE

$$\left.\begin{array}{l} A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ x = (x \ \ y)^T \end{array}\right\} \quad x'Ax = (x \ \ y) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$5xy$

$$= (x + 3y, \ \ 2x + 4y) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x(x + 3y) + y(2x + 4y)$$

$$= x^2 + 5xy + 4y^2$$

A POSITIVE DEFINITE $\longrightarrow$ ALL QUADRATIC FORMS ARE POSITIVE $\forall x \in \mathbb{R}$

VECTOR OF PORTFOLIO WEIGHTS

Consider long-only portfolio in n assets with r_i returns

w_i is the weight invested in asset i

$$\boxed{w = [w_1 \dots w_n]^T \quad \text{PORTFOLIO WEIGHTS}}$$

for a long-only portfolio

$$w_1 + w_2 + \dots w_n = 1$$

COVARIANCE MATRIX

$$V_{ij} = \mathrm{Cov}(i, j)$$

$$V = (V_{ij}) \quad n \times n \quad \text{COVARIANCE MATRIX (SYMMETRIC)}$$

- $\mathrm{cov}(i, j) = \mathrm{cov}(j, i)$
- $\mathrm{cov}(i, i) = \mathrm{Var}(i)$

PORTFOLIO AS A QUADRATIC FORM

$$\boxed{w^T V w \quad \text{VARIANCE OF THE PORTFOLIO}}$$

Being a variance it must always be positive for ANY weights

$w^T V w$ QUADRATIC FORM

SAMPLE STATISTICS AND POPULATION PARAMETERS

$x = (x_1 \dots x_n)$ RAND VAR

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \text{SAMPLE MEAN}$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad \text{VARIANCE}$$

$$s = \sqrt{s^2} \qquad \text{STANDARD DEVIATION}$$

(same units of measurement as observations)

$$s_{xy} = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \qquad \text{COVARIANCE BETWEEN } x, y$$

$$r_{xy} = \frac{S_{xy}}{S_x S_y}$$

SAMPLE CORRELATION

$$-1 \leq r_{xy} \leq 1$$

EXPECTED VALUE

$\mathbb{E}[x]$ ASSUMED OR FORECASTED EX-ANTE

$\bar{x}$ MEASURED EX-POST

VARIANCE AND STD. DEVIATION

$\mathbb{V}[x]$, σ^2 VARIANCE

$$\boxed{\mathbb{V}[x] = \mathbb{E}\left[(x - \mathbb{E}[x])^2\right]}$$

COVARIANCE

$$Cov[x,y] = \mathbb{E}\left[(x - \mathbb{E}[x])(y - \mathbb{E}[y])\right]$$

IF x, y INDEPENDENT $\longrightarrow$ $Cov(x,y) = 0$ NOT VICEVERSA

RULES FOR STATISTICAL OPERATORS

- $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$ LINEARITY
- $\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\mathbb{E}[Y] + Cov[X,Y]$
- $\mathbb{V}[aX + bY] = a^2\mathbb{V}[X] + b^2\mathbb{V}[Y] + 2ab\, Cov[X,Y]$
- $Cov(aX, bY) = ab\, Cov[X,Y]$
- $Corr(aX, bY) = sign(a \cdot b)\, Corr[X,Y]$

i.i.d RANDOM VAR

x, y are INDEPENDENT AND IDENTICALLY DISTRIBUTED if they have identical distribution and are independent.

So:

$$\mathbb{E}[x] = \mathbb{E}[y] \quad \text{and} \quad \mathbb{V}[x] = \mathbb{V}[y]$$

$$Cov[x,y] = Corr[x,y] = 0$$

NORMAL DISTRIBUTION $N(\mu,\sigma^2)$

$X \sim N(\mu, \sigma^2)$ NORMAL VAR

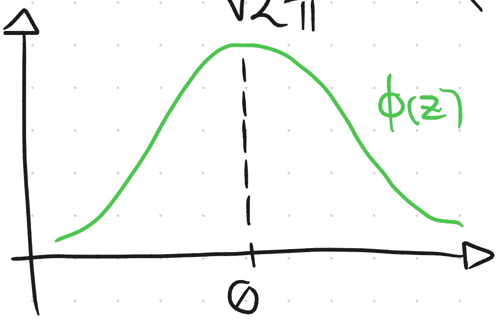
$\mu = \mathbb{E}[X], \quad \sigma^2 = \mathbb{V}[X]$

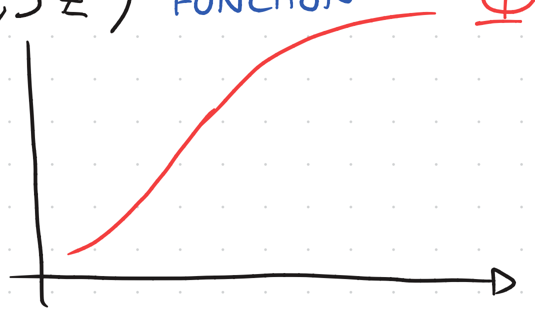
$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)}$$ DENSITY FUNCTION

$Z \sim N(0,1)$ STANDARD NORMAL VAR

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-0{,}5\, z^2)$$ DENSITY FUNCTION

$\Phi(z)$ CUMULATIVE DISTRIBUTION FUNCTION CDF



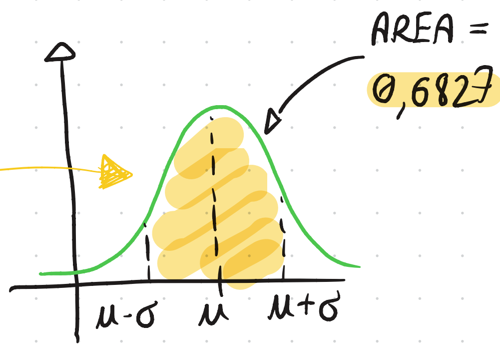


NORMAL PROBABILITIES

Normal density function is BELL-SHAPED

$$\mathbb{P}(\mu - \sigma < X < \mu + \sigma) = 0{,}6827$$

$$\mathbb{P}(\mu - 2\sigma < X < \sigma + 2\sigma) = 0{,}9545$$



QUANTILES OF NORMAL DISTRIBUTION

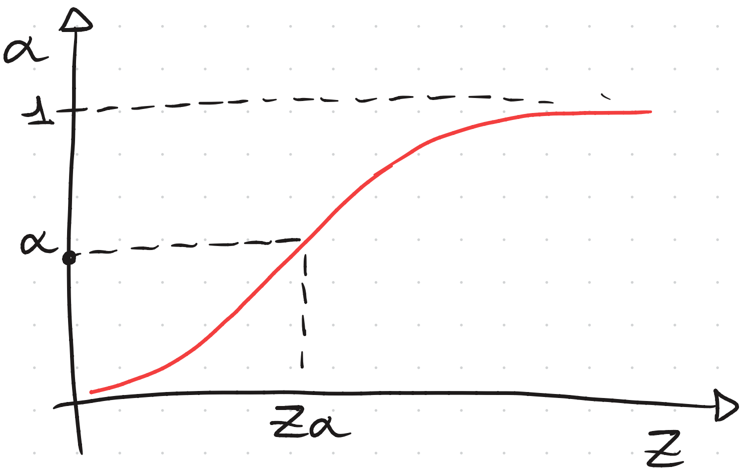
$\Phi(z) \in [0,1]$ is the distribution function for $Z \sim (0,1)$

$\phi^{-1}(\alpha)$ is the α-quantile

$\alpha \in (0,1)$

$$\boxed{\phi^{-1}(1-\alpha) = -\phi^{-1}(\alpha)}$$

$\phi^{-1}(\alpha) = Z_\alpha$ (notation)



NORMSDIST(X) $\rightarrow$ Returns $\phi(x)$

NORMSINV (P) $\rightarrow$ Returns $z \in N(0,1)$ such that $P(Z < z) = p$

SUM OF NORMAL VARIABLES

$X_1 \sim N(\mu_1, \sigma_1^2)$ with correlation ρ

$X_2 \sim N(\mu_2, \sigma_2^2)$

For any a, b

$$\boxed{aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2)}$$

STABLE PROPERTY

STANDARD NORMAL TRANSFORMATION

$X \sim N(\mu, \sigma^2)$

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

The opposite side:

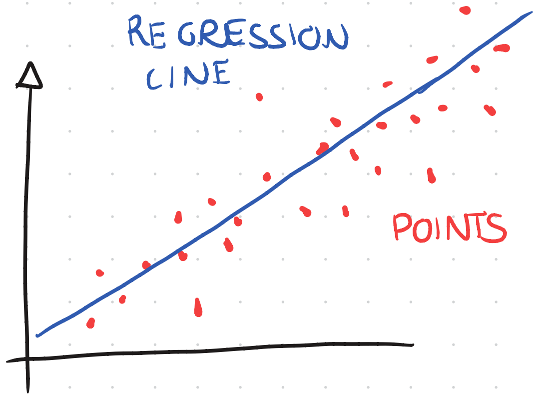
$$X = Z\sigma + \mu \sim N(\mu, \sigma^2)$$

SIMPLE LINEAR REGRESSION

SIMPLE LINEAR MODEL (SLM)

β

$\varepsilon \sim i.i.d(0, \sigma^2)$



α: INTERCEPT

β: SLOPE

ε: ERROR PROCESS

ESTIMATION

In risk analysis we use historical data

$$Y_T = \alpha + \beta X_t + \varepsilon$$

RESIDUALS

$$\boxed{\varepsilon_T = Y_T - \hat{Y}_T = Y_T - (\hat{\alpha} + \hat{\beta} X)}$$

ESTIMATED INTERCEPT & SLOPE

ESTIMATING PARAMETERS IN SLM

we minimize residual sum of squares

$$\boxed{RSS = \sum_{t=1}^{T} \varepsilon^2}$$

we derive the ordinary least squares OLS formulas

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$
$$\hat{\beta} = \frac{S_{xy}}{S_x^2}$$
$$\hat{\sigma} = \sqrt{\frac{RSS}{T-2}}$$

solving the optimization problem we end up with these formulas

T-2 because we lost 2 d.o.f for α, β

In OLS $\hat{\beta}$ is related to r_{xy} the correlation of X, Y

$$\boxed{\hat{\beta} = \frac{S_{xy}}{S_x^2} = \frac{r_{xy} S_x S_y}{S_x^2} = \underbrace{r_{xy}}_{corr}\left(\frac{S_y}{S_x}\right)}$$

β has same sign as sample correlation
β not limited between $-1, 1$

ANALYSIS OF VARIANCE (ANOVA)

Smaller RSS better for the model, but how small?
This depends on TSS

$$\boxed{TSS = \sum_{t=1}^{T} (Y_t - \bar{Y})^2}$$

TOTAL SUM OF SQUARES

$$\boxed{TSS = (T-1)\mathbb{V}(Y)}$$

(it measures variation in Y)

$$\boxed{TSS = ESS + RSS}$$

RSS: RESIDUAL SUM OF SQUARES

ESS (Explained sum of squares) is the amount of variation in Y explained by the regression

$$\boxed{R^2 = \frac{ESS}{TSS}}$$

R^2 measures the GOODNESS OF FIT of the model

MULTIPLE LINEAR REGRESSION

GENERAL LINEAR MODEL (GLM) is

$$Y_T = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \dots \beta_K X_{Kt} + \varepsilon_t$$

$\varepsilon_t \sim i.i.d\,(0, \sigma^2)$
$T = 1 \dots T$

$\beta_1 \dots \beta_K$ are the SENSITIVITIES to returns of different risk factors

MATRIX FORM

$$Y = X\beta + \varepsilon$$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad X = \begin{array}{c} K+1 \\ \overbrace{\begin{pmatrix} 1 & X_{21} & \dots & X_{K1} \\ 1 & X_{22} & \dots & X_{K2} \\ 1 & X_{23} & \dots & X_{K3} \\ 1 & & & \\ 1 & X_{2T} & \dots & X_{KT} \end{pmatrix}} \end{array} \Big\} T = \begin{pmatrix} 1 & X_2 & \dots & X_K \end{pmatrix}$$

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_T \end{pmatrix}$$

FITTED MODEL IN MATRIX FORM

We use data on Y and X to estimate params

$$\hat{Y} = \hat{\beta X}$$

$$Y = \beta X + \varepsilon$$

$$\varepsilon = Y - \hat{Y}$$ residuals

OLS fits the regression minimizing RSS

$$\boxed{RSS = \varepsilon^T \varepsilon}$$ RESIDUAL SUM OF SQUARES

$$\boxed{\hat{\beta} = (X^T X)^{-1} (X^T Y)}$$

$$\boxed{\hat{\sigma} = \sqrt{\frac{\varepsilon^T \varepsilon}{T - K}}}$$

RISK DECOMPOSITION IN THE SINGLE INDEX MODEL

$$Y_T = \alpha + \beta X_T + \varepsilon_T$$

$$\mathbb{V}(Y_T) = \underline{\beta^2}\,\underline{\mathbb{V}(X_T)} + \underline{\sigma^2}$$

3 COMPONENTS OF RISK

- PORTFOLIO SENSITIVITY (Risk relative To The market): β^2
- SYSTEMATIC RISK (nondiversifiable): $\mathbb{V}(X_t)$
- PORTFOLIO'S SPECIFIC RISK (idiosynchratic): σ^2

RISK DECOMPOSITION WITH MULTIPLE FACTORS

$$\hat{Y} = X\hat{\beta}$$

$$\boxed{\mathbb{V}(Y) = \beta^T \mathbb{V}(X)\hat{\beta}}$$

QUADRATIC FORM

EXPONENTIALS AND LOGARHYTMS

e^x

$\exp(x)$

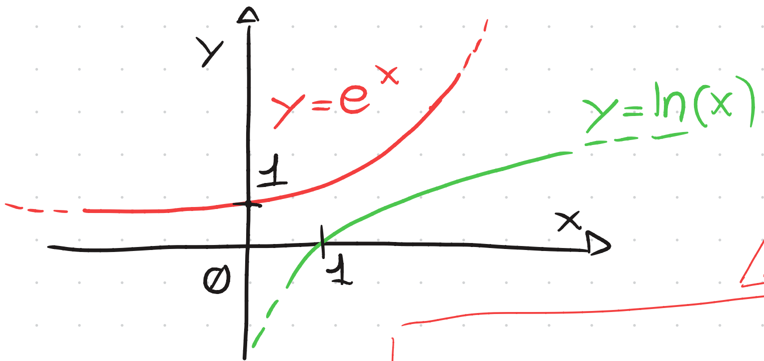
$e = 2{,}7182\ldots$ EULER'S NUMBER

$\ln(x)$

$\log_e(x)$

INVERSE FUNCTION OF $\exp(x)$

$\ln(e^x) = x$ AND $\exp(\ln x) = x$



symmetric To $y=x$

TAYLOR

EXPANSION $\ln(1+x)$

if $-1 \leq x \leq 1$:

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots}$$

if $x \to 0$:

$$\boxed{x \approx \ln(1+x)}$$

RULES FOR $\ln(x)$

- $\ln(x^\alpha) = \alpha \ln(x)$
- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

RETURNS AND VOLATILITY

P_i value of the portfolio at time t

h-day PROFIT & LOSS at time t: $P_t - P_{t-h}$

Returns only make sense if $P_t > 0$, so:

h-day return at time t: $\boxed{R_{ht} = \dfrac{P_t - P_{t-h}}{P_{t-h}}}$

when $h \to 0$, we may use LOG-RETURN to approximate R_{ht}

$$\boxed{R_{ht} = \frac{P_t - P_{t-h}}{P_{t-h}} \approx \ln\left(\frac{P_t}{P_{t-h}}\right) = \ln(P_t) - \ln(P_{t-h}) = r_{ht}}$$

using $\ln(x)$ expansion

SQUARE ROOT-OF-TIME RULE

If the returns of an asset are i.i.d and independent, then their standard deviation scales with square root of the holding period

$$\boxed{\sigma_h = \sigma_1 \cdot \sqrt{h}} \qquad \text{AND} \qquad \sigma_1 = \frac{\sigma_h}{\sqrt{h}}$$

$\to$ h = holding period

VOLATILITY

Volatility is the annualized standard deviation of h-period returns:

$$\boxed{\text{VOLATILITY} = \sigma_{250} = \sqrt{n} \cdot \sigma_n}$$

ANNUALIZED STD DEV (250 DD)

where n is the number of h-days and assuming 250 d/year

TAYLOR EXPANSION

INTUITION FOR FIRST-ORDER TAYLOR EXPANSION

$$f'(x) = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$f'(x) \cdot \varepsilon = f(x+\varepsilon) - f(x)$$

for some neighbourhood of x_0 st $\varepsilon = x - x_0$

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

TAYLOR EXPANSION

What happens when P&L is not a linear function?

Taylor provides a way to approx. the change in price, given some changes in each risk factor using a low-order polynomial.

IT is a local approximation of a NON-LINEAR function $f(x, y \ldots)$ by a polynomial in x and y.

HIGHER ORDER TAYLOR EXPANSIONS OF $f(x)$

The n^{TH} order Taylor Expansion for a n times continuously differentiable $f(x)$ is:

$$f(x+\varepsilon) - f(x) \approx \varepsilon f'(x) + \frac{\varepsilon^2}{2!} f''(x) + \frac{\varepsilon^3}{3!} f'''(x) \ldots + \frac{\varepsilon^n}{n!} f^{(n)}(x)$$

for example, the quadratic approximation is:

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0)$$

TAYLOR EXPANSION OF FUNCTIONS IN SEVERAL VARS

$$f(x+\varepsilon_x, y+\varepsilon_y) - f(x,y) \approx \varepsilon_x f_x(x,y) + \frac{\varepsilon_x^2}{2!} f_{xx}(x,y) + \varepsilon_y f_y(x,y)$$

Taylor expansion of second order to x, first to y

f_x: partial derivative of $f(x,y)$ with respect to x

f_{xx}: second partial derivative $f(x,y)$ with respect to x

CLASSIFICATION OF FINANCIAL INSTRUMENTS

A FINANCIAL ASSET is a claim on a real asset (cash, commodities...)

Examples of financial assets are SECURITIES

Two main Type of SECURITIES

- BONDS (company, governments, asset-backed securities...)
- EQUITY (shares, ETF...)

FINANCIAL INSTRUMENT

Any contract that gives rise to a financial asset of one entity and a financial liability of another entry

DERIVATIVES

- FUTURES AND FORWARDS
- OPTIONS
- SWAPS

Their Market Capitalization is ZERO ⚠

ETF: Exchange-Traded Funds

ETN: Exchange-Traded Notes

(like ETF but with maturity, subject to credit risk)

ETN do not directly own the underlying asset

They are a debt obligation of the issuer.

WHAT IS FINANCIAL RISK MANAGEMENT

RISK = UNCERTAINITY

Risk is the uncertainity in the value of a stochastic process at some time in the future

RISK HORIZON

The day at which the risk is forecast

Typical risk horizon = 1 or = 10 days

5 MAIN TYPES IN FINANCIAL RISKS

- MARKET RISK (changes in IR or asset prices)
- CREDIT RISK (default on a obligation or change in rating)
- OPERATIONAL RISK (associated with NON-financial matters)
- LIQUIDITY RISK (a transaction cannot be made)
- BASIS RISK (imperfect hedging)

ENTITIES IN FINANCIAL MARKETS

- INVESTMENT BANK
- COMMERCIAL BANK
- CORPORATE TREASURY
- INSTITUTIONAL INVESTOR
- EDGE FUND

DIVISION OF RISK-MANAGEMENT ROLES

- FRONT OFFICE (Traders, Market Makers)
- MIDDLE OFFICE (Measures and controlls the risks) limiting trading activity
- BACK OFFICE (Processes trades to comply laws and regulations)

OVERVIEW OF MARKET RISKS

- MARKET RISK

UNCERTAINTY IN THE FUTURE OF AN INVESTMENT ARISING FROM IR OR THE PRICE OF FINANCIAL MARKET.
MR MEASURED USING THE DISTRIBUTION OF P&L
SOMETIMES WE ASSUME that P&L $\sim N(0, \sigma^2)$

- EQUITY RISK

EQUITY RISKS IS THE UNCERTAINTY DUE TO CHANGES IN EQUITY RISK FACTORS

- CURRENCY RISK

UNCERTAINTY DUE TO FLUCTUATIONS IN EXCHANGE RATES

- INTEREST RATE RISK

UNCERTAINTY DUE TO CHANGES IN IR
FIXED AND FLOATING INCOME PORTFOLIO (CASH FLOW PORTFOLIO) CAN HAVE LARGE IR RISK.

- MARKET RISK FACTOR

A BROAD, MARKET-WIDE INDEX THAT CAPTURES THE OVERALL MARKET RISK

RISK FACTOR MAPPING

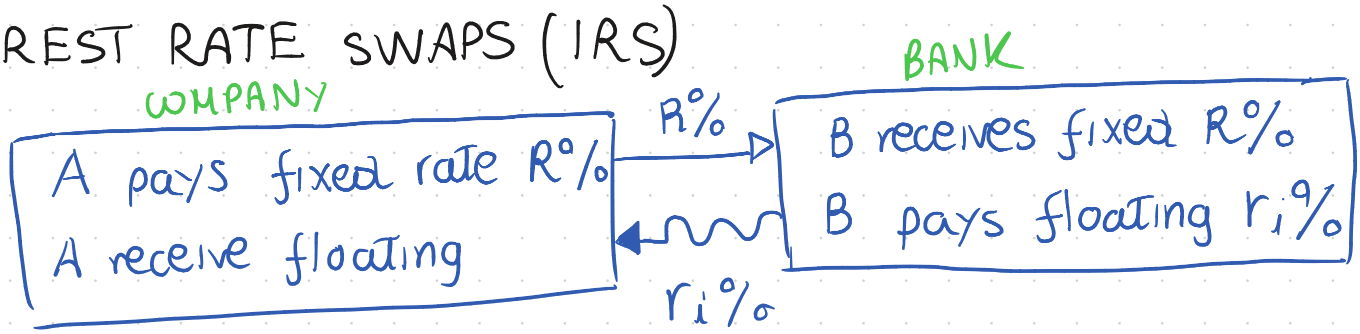
ANY RISK FACTOR SENSITIVITY MEASURE UNIT OF CHANGE IN THE PORTFOLIO FOR EVERY UNIT OF CHANGE IN THE RF.
LINEAR REGRESSION ⟶ SENSITIVITY (β)

RISK FACTORS FOR A DOMESTIC BOND P.

Suppose a portfolio of UK Bonds with various coupon/maturities over 10 yrs.
Each bond is a series of CF
The risk factors are a TERM STRUCTURE OF IR
The sensitivities are called PRESENT VALUE OF A BASIS POINT (PV01), PVBP. (change in bond price for every 0.01 of IR)

INTRODUCTION TO CREDIT RISK AND INTEREST RATE SWAP

INTEREST RATE SWAPS (IRS)



R% fixed rate

r_i% floating rate, usually linked to LIBOR + spread

INTEREST RATE SWAPS

Two counterparties (typically bank and company) in a IRS swap fixed payments R% for floating payments r_i%. on a notional amount, up to a certain maturity with payments at $t = 1, 2 \dots N$

r_i% usually linked to LIBOR + spread

Swap Rate R% is fixed so NPV of cash flows is zero

Fixed rate cash flows are known ⟶ NOT RISKY

Floating rate cash flows ⟶ UNCERTAINTY

MARTET-TO-MARKET ACCOUNTING

Banks use M-T-M accounting. Asset/Liab. are accounted "every day" at market price

Future cash flows are discounted at a rate linked to LIBOR (usually LIBOR + SPREAD)

So the only risk factor is the amount linked to spreads.

LIBOR + Spread rate 1 ⟶ Determines cash flow

LIBOR + Spread rate 2 ⟶ Determines discount rate

RISK FACTOR : difference between spread rate 1 and 2.

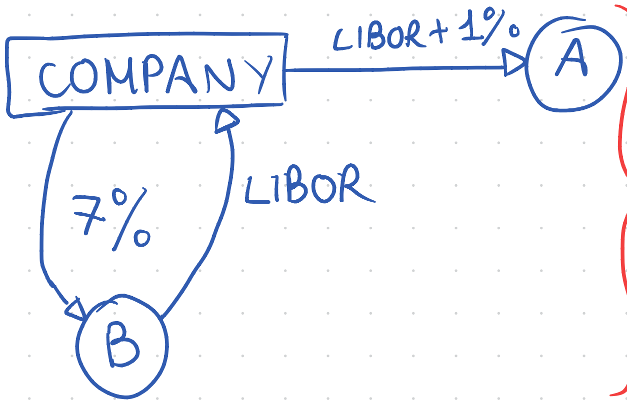
Under MTM, usually fixed CF are more risky.

- A bank prefers to give floating rate loans to companies (perceived as less risky)
- Companies using cash accounting prefer fixed-rate loans.

EXAMPLE

A company gets 8-y loan for \$1m from Bank (A) on which it pays LIBOR + 1%.

The company enters a 8-y IRS with Bank (B) with notional \$1m to receive LIBOR and pay a fixed swap rate of 7%



By matching the durations and the notional, the company is just paying a 8% FIXED RATE LOAN

NO LIBOR EXPOSURE

TYPES OF CREDIT RISKS

UNCERTAINTY ARISING FROM A CREDIT EVENT

- SPREAD RISK: change in credit rating ⟶ change in credit spread (also known as migration risk)
- DEFAULT RISK

Credit spread is the extra premium required by the market for taking credit exposure.

credit default occurs when a counterparty does not meet the obligations

Issuer risk: the issuer defaults on the principal/interest to the creditor (obligee)

RECOVERY RATE: % of outstanding payments recovered

seniority structure to determine different recovery rates.

CREDIT RATINGS

RATING AGENCIES (MOODY'S, S&P, FITCH) → RATE/GRADE representing possibilities to default (AAA, AA, A, BBB, BB, B, CCC, CC, C)
AAB Best

Rating agencies use historical data to analyse credit migration and transition (changes in ratings during a period)

credit transition → migration frequencies → transition probabilities → CREDIT TRANSITION MATRIX

EXAMPLE OF 1-YEAR TRANSITION MATRIX

original rating	probability of migrating to rating by year end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	93.66	5.83	0.40	0.08	0.03	0.00	0.00	0.00
AA	0.66	91.72	6.94	0.49	0.06	0.09	0.02	0.01
A	0.07	2.25	91.76	5.19	0.49	0.20	0.01	0.04
BBB	0.03	0.25	4.83	89.26	4.44	0.81	0.16	0.22
BB	0.03	0.07	0.44	6.67	83.31	7.47	1.05	0.98
B	0.00	0.10	0.33	0.46	5.77	84.19	3.87	5.30
CCC	0.16	0.00	0.31	0.93	2.00	10.74	63.96	21.94
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

CREDIT DERIVATIVES

CREDIT DERIVATIVE

Instrument/Technique designed to separate the credit risk of a company and transfer it to another entity.

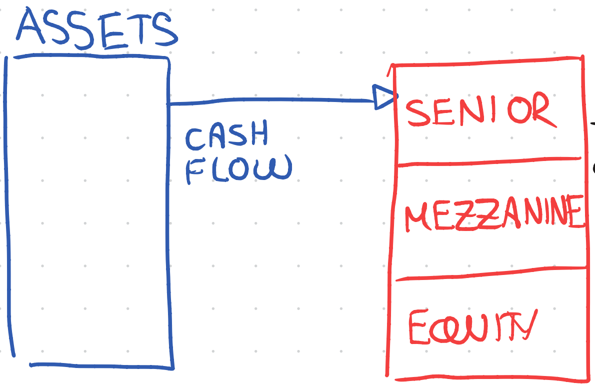
- A funded credit derivative is backed by some assets
  - ABS (asset-backed securities)
  - CDO (collaterized debt obbligation)
- An unfunded credit derivative is sold without protection
  - CDS (credit default swaps)

ASSET-BACKED-SECURITY

- Payments of the security come from a pool of underlying assets (small and illiquid assets, often not sold individually)
- Polling assets into financial instroments → SECURIZATION

COLLATERALIZED DEBT OBLIGATION (CDO)

- A CDO is a structured ABS which pays investors in a prescribed sequence based on the CF the CDO collects
- A CDO allows investors to take different credit risk according to their appetite ⟶ DIFFERENT TRANCHES

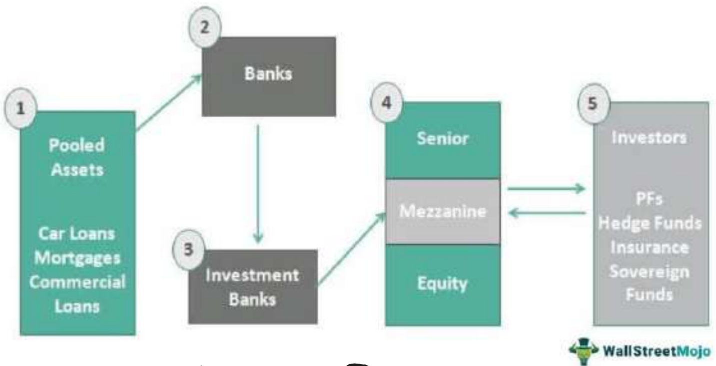


AAA, safest tranches, they are paid first from the CF, priority in case of default

riskiest, absorbing losses if underlying assets fail

Different investors buy different tranches

Collateralized Debt Obligation (CDO)



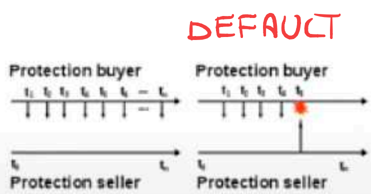
CREDIT DEFAULT SWAP

The buyer of the CDS makes a series of payments to the seller based on the CDS spread, in exchange he receives a pay-off from the seller if the credit defaults (seller ≠ credit, two distinct parties)

It allows to transfer the default credit risk from the buyer to the seller (≈ INSURANCE)

Left: The buyer purchases a CDS at time t_0 and makes regular premium payments at times $t_1, t_2, t_3, \ldots$ and so on until the end of the contract unless the associated instrument suffers a credit default

Right: If the underlying instrument suffers a credit default at t_5, then the seller compensates the buyer for that loss, and the buyer ceases paying premiums to the seller



NAKED CDS

A CDS where the buyer doesn't own the asset that the CDS is referencing ⟶ The buyer is speculating on the creditworthiness of the asset without having it.

≈ buying an insurance on a house you don't have, hoping that will burn down to receive the payoff.

INTRODUCTION TO VOLATILITY

VOLATILITY

Two types of volatility:

- IMPLIED VOLATILITY: implicit in the price of a vanilla option. It's derived from the market price of the option using the Black Sholes formula.
- STATISTICAL VOLATILITY: calculated from time-series returns

Historical volatility is a statiscal volatility calculated from the standard deviation of n historical returns and the annualized.

HISTORICAL VOLATILITY FROM A ROLLING WINDOW

Step-by-step procedure:

- Fix window size n
- Take $N \gg n$ sample returns
- Calculate volatility of $[1, n]$ returns
- Shift by one
- Calculate volatility of $[1+1, n+1]$ returns
- Repeat until you did $[N-n, N]$

VOLATILITY OF P&L VS VOLATILITY OF RETURNS

Instead of using returns, ofthen P&L volatility is used to measure risk

$$\boxed{\text{P\&L VOLATILITY} = \text{returns volatility} \times \text{current portfolio value}}$$

EXPONENTIALLY WEIGHTED MOVING AVERAGE (EWMA)

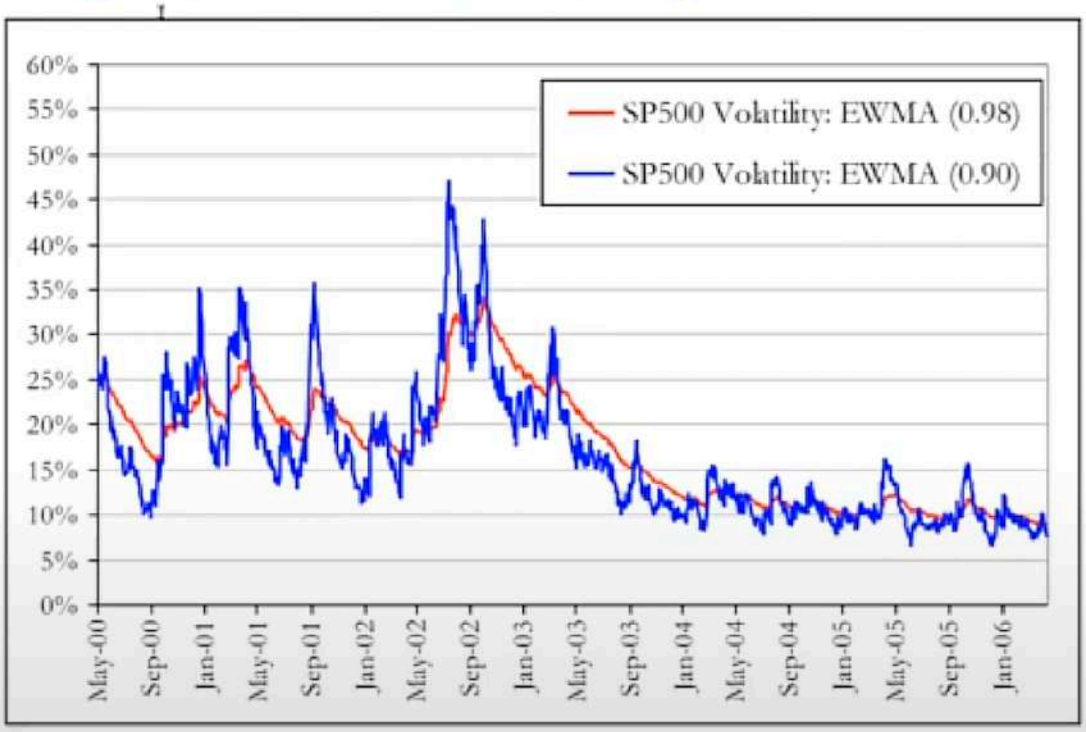
$$\boxed{\sigma_t^2 = (1-\lambda)\sum_{i=1}^{\infty} \lambda^{i-1}\, r_{t-1}^2}$$

exponential decay at rate y

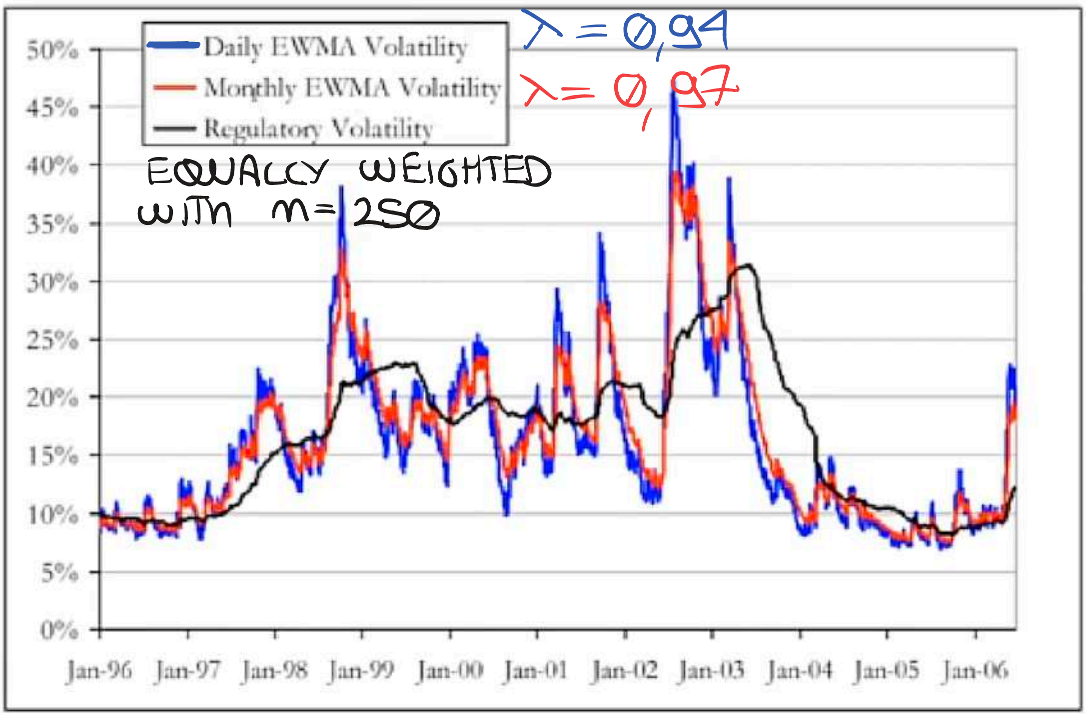
$$\boxed{\sigma_t^2 = (1-\lambda)\, r_{t-1}^2 + \lambda^2 \sigma_{t-1}^2}$$

recursive formula

Higher λ gives a smoother volatility estimate



RiskMetrics Data



EWMA estimate is a forecast for ALL horizons.
EWMA's assumption is still that returns are i.i.d.
EWMA can be extended to include the ASSYMETRIC VOLATILITY RESPONSE

$$\sigma_t^2 = (1-\lambda)(r_{t-1} - \psi)^2 + \lambda \sigma_{t-1}^2$$

ψ : ADDITIONAL PARAMETER

LINEAR PORTFOLIOS AND THEIR VOLATILITY

P&L is a linear function of the results of their assets.

$$r_p = w' r = \sum_{i=1}^{n} w_i r_i$$

$$\mathbb{E}[r_p] = w' \mathbb{E}[r] = \sum_{i=1}^{n} w_i \mathbb{E}[r_i]$$

$$\sigma_p^2 = \sum w_i \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j>i} w_i w_j \sigma_{ij}$$

$$= \sum w_i \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j>i} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

$$\text{Corr} = \frac{\text{Cov}(x,y)}{\text{Var}(x)\text{Var}(y)}$$

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

$$\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$$

COVARIANCE MATRIX

$$V = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \dots & \sigma_{1n}^2 \\ \sigma_{21}^2 & & & \vdots \\ \vdots^2 & & & \\ \sigma_{n1} & & \dots & \sigma_{nn}^2 \end{pmatrix}$$

covariance matrix of n assets

$$\sigma_p^2 = w^T V w$$

Portfolio volatility (annualized std. dev) can be calculated in 2:

- using $\sigma_p^2 = w^T V w$ and annualizing $\sigma_{250} = \sqrt{n}\,\sigma_n$
- constructing $r_p = w^T r$ and then measuring

THE EQUITY BETA

SINGLE INDEX MODEL

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

$\varepsilon_t \sim i.i.d.(0, \sigma^2)$

EQUITY BETA β measures the sensitivity of the stock to the variations of the index.

(Market neutral strategies aim to reach $\beta \to 0$)

RISK DECOMPOSITION

$$\boxed{\mathbb{V}(Y_t) = \beta^2 \mathbb{V}(X_t) + \sigma^2}$$

Three components of risk

- SENSITIVITY β^2
- SYSTEMATIC / UNDIVERSIFIABLE RISK $\mathbb{V}(X_t)$
- SPECIFIC / IDIOSYNCHRATIC / DIVERSIFIABLE RISK σ^2

(specific risk $\to 0$ for n assets $\to \infty$)

BETA IN $

$$\beta = \frac{S_{yx}}{S_x^2} = \frac{r_{xy} S_x S_y}{S_x^2} = \underbrace{r_{xy}}_{\text{corr}} \frac{S_y}{S_x}$$

(formula from SLM optimization)

⚠ Long/short portfolio can take zero or negative returns → so returns do not exist → so we use P&L

$$\boxed{\beta^{\$}_{xy} = r^{\$}_{xy} \frac{S^{\$}_y}{S^{\$}_x}}$$

corr. between Portfolio P&L and Index P&L

$$\boxed{\beta^{\%}_t = \frac{\beta^{\$}_t}{\text{PORTFOLIO VALUE}}}$$

PORTFOLIO BETA

$$\beta_P = \sum_{i=1}^{m} w_i \beta_i = w^T \beta$$

$$\beta = \frac{\sigma_{PX}}{\sigma_X^2}$$

σ_{PX} = Covariance between portfolio and index returns

σ_X^2 = Valatility of index returns

VALUE AT RISK BASICS

VAR

Value at Risk (VaR) is a loss we are confident will NOT exceed if the current portfolio is NOT rebalanced on a defined risk horizon

$VaR_{h,\alpha}$ is the $\alpha\%$, h-day VaR wich is $-1 \times \alpha$ - quantile of the discounted h-day P&L distrib.

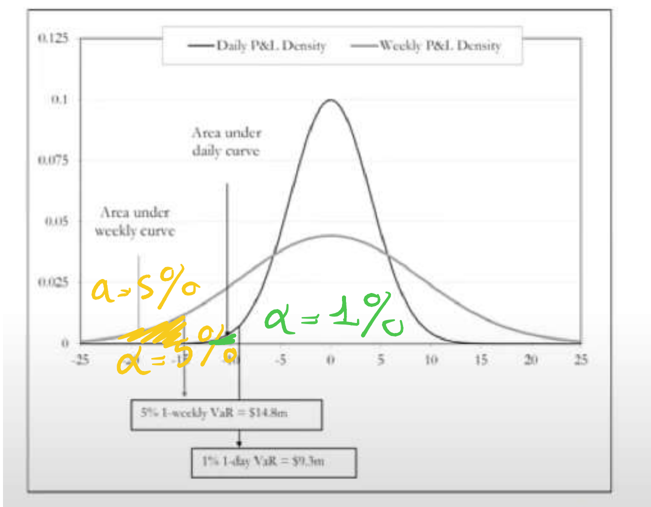
α : SIGNIFICANCE LEVEL of VaR estimation

$1-\alpha$: CONFIDENCE LEVEL

VaR:

- TIME HORIZON h
- CONFIDENCE LEVEL (prob that loss won't exceed)
- POTENTIAL LOSS AMOUNT

$$\boxed{\mathbb{P}(P\&L < \underbrace{-Var_{\alpha,h}}_{\text{Potential Loss}}) = \alpha}$$



⚠ If LONG-SHORT Portfolio:

VaR in \$ = VaR in % × current Portfolio Value

DEFINITION OF PARAMS

$h = 10$ market risk capital requirement

$h = 1$ backtesting VaR model

VAR MODELS

4 STEPS FOR BUILDING VaR ESTIMATION

1. set parameter h
2. create probability distribution for discounted returns (or P&L) over next h days
3. set α significance level
4. Estimate VaR as $-1 \times \alpha$-quantile of the distrib.

⚠ Step 2 makes the difference between VaR models!

- Bank regulations recommend 3-5 years historical data to estimate P&L distribution.

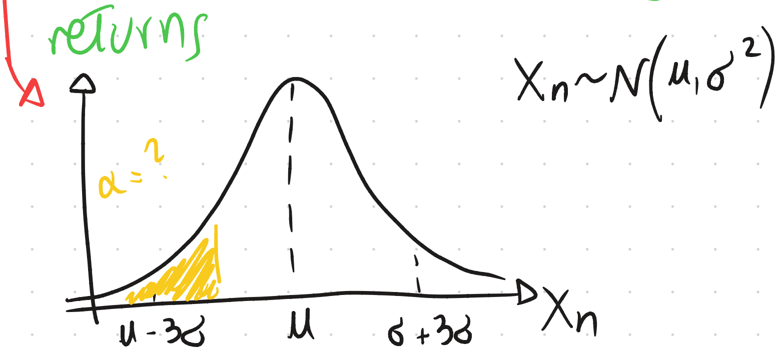
HOW TO ESTIMATE DISTRIBUTION

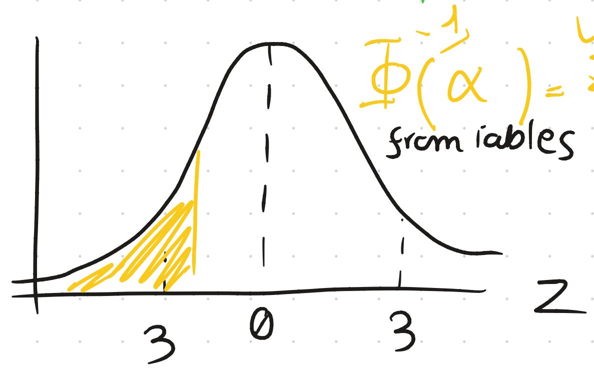
- NORMAL VAR: assumes $N(\mu, \sigma^2)$ using μ, σ^2 of returns or P&L from historical data

⚠ NORMAL VAR APPLIES ONLY TO LINEAR PORTFOLIOS!!!

- HISTORICAL VAR: Build histogram of historical returns and read from the quantile
- MONTE CARLO VAR: Assumes that returns have some parametric distribution, compute simulations and then read VaR as quantile

Use the =PERCENTILE function on the historical/simulated returns





STANDARD NORMAL TRANSFORMATION (NORMAL VAR)

$$\Phi^{-1}(\alpha) = \tilde{Z}$$

$$\Phi(\tilde{Z}) = P(Z < \tilde{Z}) = \alpha$$

$$P(Z < \tilde{Z}) = P\left(\underbrace{\frac{Z+\mu}{\sigma}}_{N\sim(\mu,\sigma^2)} < \frac{\tilde{Z}+\mu}{\sigma}\right) = \alpha$$

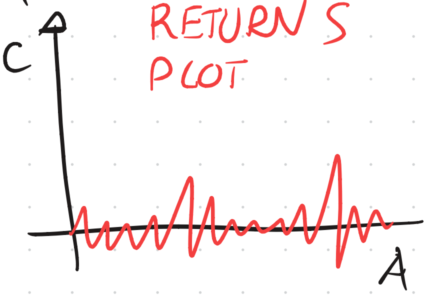
VaR with $1-\alpha$ confidence level

HISTORICAL & MONTECARLO VaR

A_Date B_PortfolioValue C_Returns

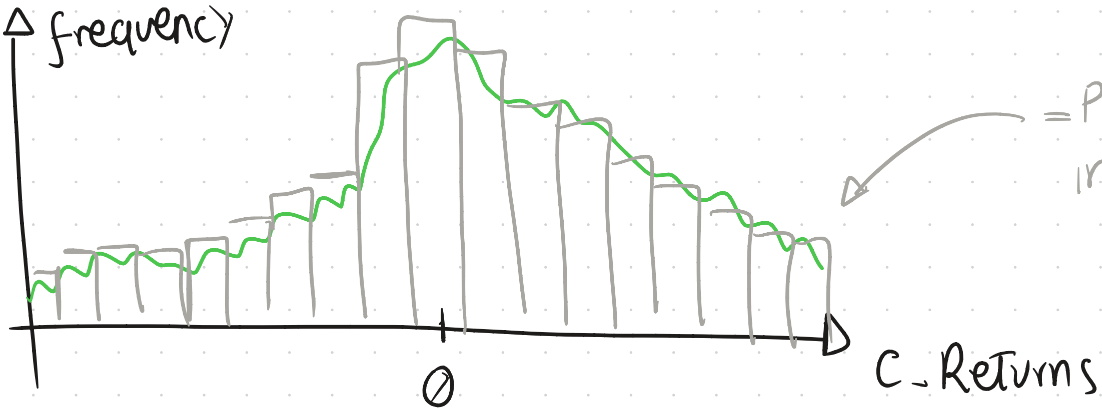
$(B_n - B_{n-1})/B_{n-1}$

TIME-SERIES RETURNS PLOT



For MonteCarlo VaR returns are SIMULATED, not historical

RETURNS PLOT _ HISTOGRAM



= PERCENTILE in Excel

NORMAL VAR MODEL

Assumption That h-day returns are $X_h \sim N(\mu_h, \sigma_h^2)$
Then

$$\boxed{\begin{aligned}\%\,VaR_{h,\alpha} &= \Phi^{-1}(1-\alpha)\,\sigma_h - \mu_h \\ &= -\Phi^{-1}(\alpha)\,\sigma_h - \mu_h\end{aligned}}$$

PROOF

$$P(X_h < X_{h,\alpha}) = P\left(Z < \frac{X_{h\alpha} - \mu_h}{\sigma_h}\right) = \alpha$$

By definition

$$P(Z < \Phi^{-1}(\alpha)) = \alpha$$

So we obtain:

$$\frac{x_{\alpha h} - \mu_h}{\sigma_h} = \Phi^{-1}(\alpha) = -\Phi^{-1}(1-\alpha)$$

$$\%VaR_{h,\alpha} = -x_{h,\alpha} = \Phi^{-1}(1-\alpha)\sigma_h - \mu_h)$$ normal VaR

CONVERTIN NORMAL VAR TO \$ VAR

$$\boxed{\$VaR_{h,\alpha} = P_t \cdot \left(\Phi^{-1}(1-\alpha)\sigma_h - \mu_h\right)}$$

where P_t portfolio value

$$\boxed{\$VaR_{h,\alpha} = \Phi^{-1}(1-\alpha)\sigma_h^{\$} - \mu_h^{\$}}$$

where $\mu_h^{\$}$ and $\sigma_h^{\$}$ are mean and std of P&L

HISTORICAL VAR

Two ways for getting historical data:

- NO REBALANCING n shares in each asset are costant
  weights change over time
  (VaR is different to scale at ≠ risk horizons)
- REBALANCING weights are costant

VAR FROM HISTORICAL DATA

IT's common to use 1-day historical VaR and scale using "square root of time

MONTECARLO VAR

α% h-day VaR, expressed as % of the current portfolio value, is (minus) the α-quantile of simulated h-day discounted return distribution

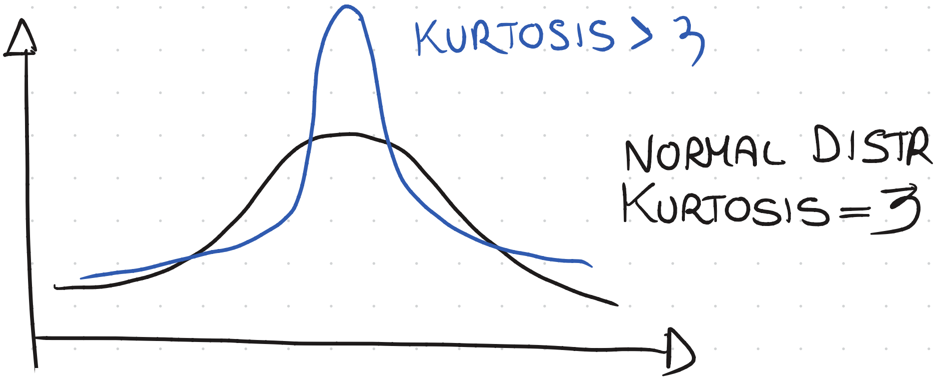
- often more reliable than the historical VaR for low α and large h

To simulate observation with distribution $F(x)$:

- Pick random $u \in Uniform(0,1)$, plug u into $F^{-1}(u) = X$

COMPARISON OF VAR MODELS

- VaR increases with confidence level $(1-\alpha)$ and risk horizon
- 1% 10-d VaR of S&P500 is $\approx 10\%$
- Usually 1% historical VaR > 5% historical VaR due to excess kurtosis



Advantages and Limitations of Different VaR Models

Normal VaR

- Advantage: VaR may be calculated using an easy formula
- Limitation: Only applies to portfolios that are a linear function of normally distributed risk factors

Historical VaR

- Advantage: No parametric assumption about returns distribution, applies to any portfolio
- Limitation: Sample size needs to be large for accuracy in tails

Monte Carlo VaR

- Advantage: Applies to any portfolio
- Limitation: Large number (e.g. 10^6) simulations $\Rightarrow$ time consuming

SCALING VAR

Using the log-return approximation for ordinary returns log returns are i.i.d so:

$$r_{ht} = \sum_{i=0}^{h-1} r_{1,t+i}$$

The mean of h-day returns is h-times the mean of 1-day return, same applies to variance.

$$\mu_h = \mu_1 \cdot h$$

$$\sigma_h^2 = \sigma_1^2 \cdot h \longrightarrow \sigma_h = \sigma_1 \sqrt{h}$$

So we obtain

$$\boxed{\% VaR_{h,\alpha} = \Phi^{-1}(1-\alpha)\sigma_1\sqrt{h} - \mu_1 h}$$

FIRST ORDER AUTOREGRESSIVE MODEL - AR(1)

⚠ Financial Assets are not usually i.i.d

(This doesn't matter until $h \leq 10$, for larger h we should consider the AUTOCORRELATION in returns)

Therefore suppose:

$$\boxed{\begin{aligned} r_t &= a + \rho r_{t-1} + \varepsilon_t \\ \varepsilon_t &\sim N(0, \sigma^2) \end{aligned}} \quad AR(1)$$

ρ denotes the correlation in the returns

if $\rho = 0 \longrightarrow$ RETURNS ARE INDEPENDENT

SCALING VaR WITH AR(1)

The h-day return is still the sum of 1-day returns:

$$r_h = \sum_{i=0}^{h-1} r_{1,t+i}$$

The variance over period h is however given by

$$\sigma_h = \sqrt{\tilde{h}}\,\sigma_1$$

$$\tilde{h} = h + 2\varrho(1-\varrho)^{-2}\left((h-1)(1-\varrho) - \varrho(1-\varrho^{h-1})\right)$$

The mean still scales as

$$\mu_h = \mu_1 \cdot h$$

VaR with AR(1) model is therefore calculated with these μ_h, σ_h

LINEAR EQUITY PORTFOLIOS AND THEIR MARKET-RISK FACTORS

Any domestic stock portfolio is LINEAR

$$R_p = w^T r = \sum_{i=1}^{n} w_i R_i$$

DOMESTIC PORTFOLIO = shares in the investor currency, no FX risk

$$\beta_p = w^T \beta = \sum_{i=1}^{n} w_i \beta_i$$

Multiple Risk Factors:

$$Y_t = \alpha + \beta_1 X_{1t} + \beta_2 X_{2t} + \dots + \beta_n X_{nt} + \varepsilon_t$$

SUBADDITIVITY: TOTAL VARs < SUM OF VARS

Total VaR take into account of diversification which aries because risk factors have correlation ≤ 1

In general:

$$\boxed{\text{Sum of the any components of the VaRs} \geq \text{Total VaR of the portfolio with these Risk Factors}}$$

EQUITY VAR

SPECIFIC / IDIOSYNCRATIC VAR

Consider a linear portfolio

$Y_t = \alpha + \beta X_t + e_t$ RESIDUAL

e_t allows to quantify the specific / idiosyncratic risk using variance of residuals (s^2 variance of e_t)

$$\boxed{\text{Specific VaR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\, s\sqrt{h}}$$

(assuming X_t normally distributed)

EQUITY / SYSTEMATIC VAR

$$\boxed{\begin{aligned}\text{Equity VaR}_{h,\alpha} &= \hat{\beta} \times \text{Market VaR}_{h,\alpha} = \\ &= \hat{\beta}\left(\Phi^{-1}(1-\alpha)\sigma_h - \mu_h\right)\end{aligned}}$$

where $\mu_h = h \cdot \mu_1$ and $\sigma_h = \sigma_1\sqrt{h}$

DECOMPOSITION OF VaR

The rule for a variance sum (assuming ZERO covariance) implies:

EQUITY VAR + SPECIFIC VAR = TOTAL VAR

given by $\beta^2 Var(X)$ given by σ^2 of E_t

Hence, in the NORMAL VAR MODEL

$$\boxed{\sqrt{(\text{SPECIFIC VAR})^2 + (\text{EQUITY VAR})^2} = \text{TOTAL VAR}}$$

EQUITY VAR WITH MULTIPLE RISK FACTORS

Suppose there are n different risk factors

$$\hat{Y}_t = \hat{\beta}^T X_t = \sum_{i=1}^{n} \hat{\beta}_i X_{ti}$$

$\hat{\beta}_i$ are OLS (ORDINARY LEAST SQUARE) estimates for a multiple linear regression

SYSTEMATIC VAR

$$\mathbb{V}(\hat{Y}) = \sum_{i=1}^{n} \hat{\beta}_i^2 \mathbb{V}(X_i) + 2 \sum_{i=1}^{n} \sum_{j>i} \hat{\beta}_i \hat{\beta}_j \operatorname{Cov}(X_i, X_j)$$

$$= \hat{\beta}^T V \hat{\beta}$$ as a quadratic form

where V is the covariance matrix of the Risk Factors Returns

Typically we use daily or weekly returns for RISK MAPPING

- daily returns h-day Cov-Matrix $= hV$
- weekly returns h-day Cov-Matrix $= \frac{h}{5} V$

Assuming now $\hat{Y}_t$, X_t daily returns with $\mu_h = 0$:

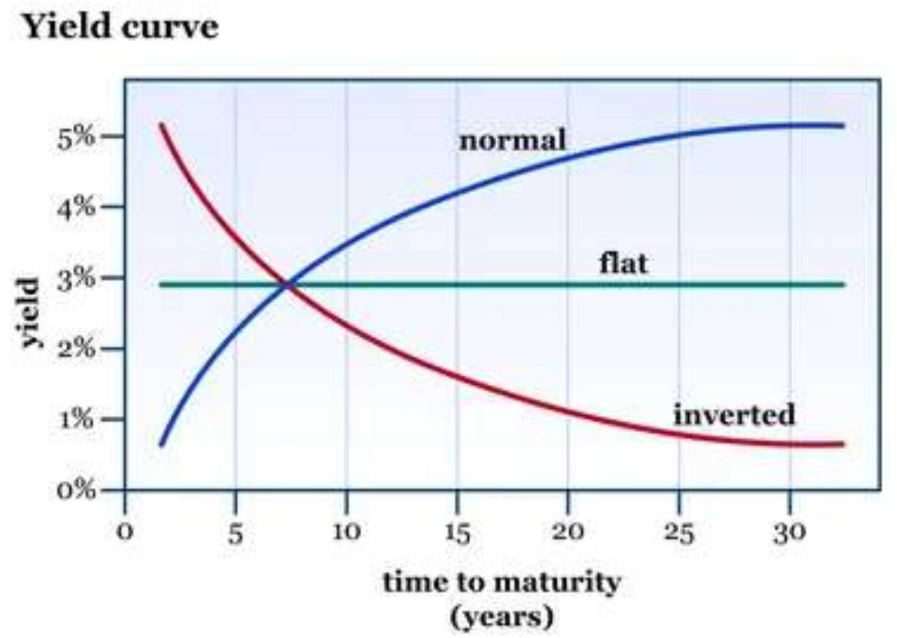
$$\boxed{\text{EQUITY } VAR_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{h\, \hat{\beta}^T V \hat{\beta}}}$$

VaR determined by different risk factors

depending on daily or weekly returns

CASH-FLOW PORTFOLIOS AND THEIR RISK FACTORS

Financial instruments represented by cash flows: loans, IR swaps, OTC agreements, bonds...
Future value of CF depends on the DISCOUNT RATE (for some cash flows – IRS or loans) the future value depends on the IR in the agreement
The YIELD CURVES are the risk factors!!!

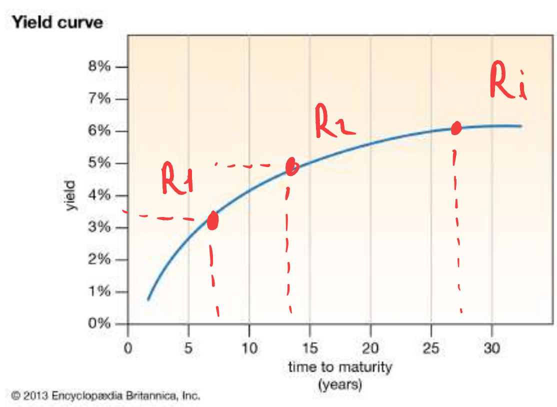


MAPPING CASH FLOWS

Risk Factors are changes in IR
Given a yield curve, we take a subset of rates at some fixed vertices

EXAMPLE: PORTFOLIO OF BONDS with maturities up to 25 years
$\{T_1, T_2 \dots T_{25}\}$ at $\{1, 2 \dots 25\}$ years
In this case, we therefore denote INTEREST RATES r as

$$r = (R_1, R_2 \dots R_n)^T$$



BASIS POINTS

$\Delta r = (\Delta R_1, \Delta R_2 \dots \Delta R_n)^T$ Typically measured daily, weekly, monthly

Δr is measured in BASIS POINTS (including his std dev)

THE CASH FLOW MAPPING PROBLEM

suppose a cash flow i falls between interest rates at T_1 an T_2?

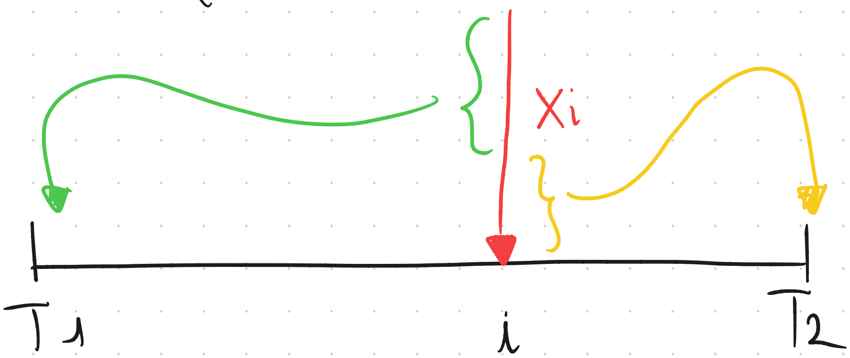
⚠ How can we divide the CF between T_1 and T_2 to keep the same present value and to keep == volatility before and after the mapping?

PRESENT VALUE (PV) OF A CV

The PV x_i of a CF C_i at maturity i with IR R_i is

$$\boxed{x_i = C_i (1 + R_i)^{-i}}$$

Consider now PV of x_i at some vertex i beteween $[T_1, T_2]$
Then, if p is the quantity mapped to T_1 $(0 \leq p \leq 1)$ and $(1-p)$ is mapped in T_2, the PV will be unchanged



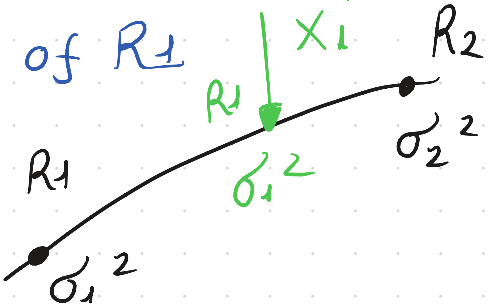
Any choice $0 \leq p \leq 1$ PV will be the same, before and after the mapping.

To also keep variance unchanged by the mapping, we also need:

$$\sigma_1^2 p^2 + \sigma_2^2 (1-p)^2 + 2\rho_{12}\sigma_1\sigma_2 p(1-p) = \sigma_i^2$$

where σ_i^2 variance of R_i, σ_1^2 variance of R_1

Solve for p



VAR MODELS FOR CASH FLOWS PORTFOLIO

Risk factor sensitivities are the change in PV of the cf when interest rates increases by 1 basis point.
It's called PRESENT VALUE of a basis point move (PV01)

PV01

Given a cf C with interest rate R and maturity T

$$\boxed{PV01 = C\left((1+R+0{,}01\%)^{-T} - (1+R)^{-T} \right)}$$

$$C > 0 \longrightarrow PV01 < 0$$

In a large mapped CF portfolio, $PV01_i$ is the change of the PV at vertex T_i when R_i increases by 1 BP
So a portfolio is mapped by a PV01 vector

$$p = (PV01_1, PV01_2 \dots PV01_n)^T$$

The factor model reprents the change in value as:

$$\boxed{\Delta P_t = PV01_1 \Delta R_1 + \dots PV01_n \Delta R_n = p' \Delta r_T}$$

This is a exact-linear model - NOT a repression

VARIANCE OF A CASHFLOW PORTFOLIO

Considering Δr daily changes, Taking $\Delta P = p' \Delta r$

$$\boxed{\mathbb{V}[\Delta P] = p' \mathbb{V}[\Delta r] p}$$

$\mathbb{V}[\Delta r]$ is the cov. matrix of daily changes in IR

VAR FOR CASHFLOWS

$$\boxed{VAR_{1,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{\mathbb{V}[\Delta P]}}$$

(using daily changes)

For any frequency of the data, Var can be scaled using the square-root-of time rule as usual

BASIC OPTION THEORY

Calls and puts are bets on stochastic "underlying"

(The underlying is anything measurable – stock price, exchange rate, a temperature ecc ecc)

- CALL: Right to buy the underlying
- PUT: Right to sell the underlying

They DO NOT have to be exercised

VANILLA OPTIONS

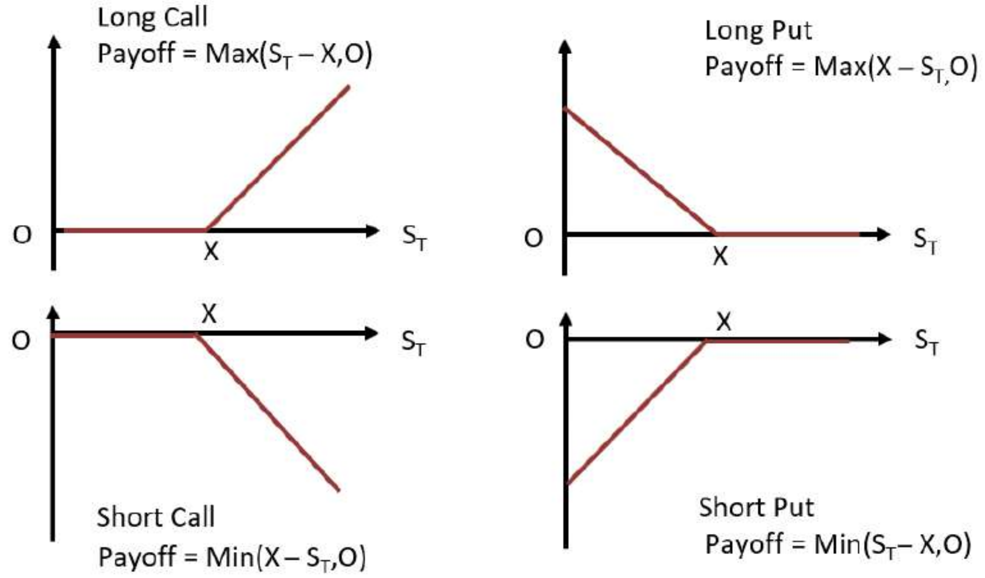
A standard European call or put

- K strike price
- T maturity

K is the price you pay if you exercise the option at time T

$$\boxed{\begin{aligned} &\text{CALL PAY OFF} = \max\{S_T - K, 0\} \\ &\text{PUT PAF OFF} = \max\{K - S_T, 0\} \end{aligned}}$$

for long position



MONEYNESS OF OPTIONS

Most Tradings focus on ATM (aT-The-money) and midTerm OTM (ouT-of-The-money)

MONEYNESS

	ITM (usually expensive)	ATM	OTM
CALL	$S_t > K$ $M_t > 1$	$S_t = K$ $M_t = 1$	$S_t < K$ $M_t < 1$
PUT	$S_t < K$ $M_t < 1$	$S_t = K$ $M_t = 1$	$S_t > K$ $M_t > 1$

- Very low K puT options on a sTock/index works as insurance ⟶ Demands rises when crisis/crash are expected ...

BuT model prices for opTion are based on The evoluTion of the underlying.

RISK NEUTRAL VALUATION RNV)

ASSUMPTIONS

- Log reTurns are i.i.d wiTh NORMAL DISTRIBUTION
- Expected ToTal reTurn is equal To The risk-free rate

⟶ ASSUMPTIONS OF THE GEOMETRIC BROWNIAN MOTION (GBM) under The RISK-NEUTRAL MEASURE

Therefore, in The GBM, There is a UNIQUE model price
All The vanilla opTions have The same value IRRESPECTIVE of The risk preferences.

↳ Under The RNV The price of an opTion is derived from The expected payoff under risk-neuTral measure

THE BLACK-SCHOLES MODEL

GMB Assumptions:

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left((\underset{\substack{\uparrow\\ \text{RISK FREE}}}{r} - \underset{\substack{\uparrow\\ \text{ADJUSTMENT FOR DIVIDENDS or...}}}{y})t, \sigma^2 t\right)$$

This occurs when S_t follows the GBM stochastic process

$$\frac{dS_t}{S_t} = (r-y)dt + \sigma d\underbrace{W_t}$$

W_t is a WEINER PROCESS

The moneyness of an option is usually given by

$$\boxed{M_t = \frac{S_t e^{-y(T-t)}}{K e^{-r(T-t)}}}$$

MONEYNESS OF AN OPTION

Equiventually, we could use $m_t = \ln(M_t)$

BLACK-SCHOLES FORMULA

CALL OPTION:

$$C_t = e^{-y(T-t)} S_t \Phi(d_{1t}) - e^{-r(T-t)} K \Phi(d_{2t})$$

PUT OPTION

$$P_t = -e^{-y(T-t)} S_t \Phi(-d_{1t}) + e^{-r(T-t)} K \Phi(-d_{2t})$$

where

$$d_1 = \frac{m_t}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$$

$$d_{2t} = \frac{m_t}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}$$

INTERPRETATION OF BLACK-SCHOLES

Setting $r = y = 0$:

$$C_t = S_t \Phi(d_{1t}) - K\Phi(d_{2t})$$
$$P_t = K\Phi(-d_{2t}) - S_t\Phi(-d_{2t})$$

- ITM CALL, OTM PUT

$$\left.\begin{array}{l}\Phi(d_{1t}) \approx \Phi(d_{2t}) \longrightarrow 1 \\ \Phi(-d_{1t}) \approx \Phi(-d_{2t}) \longrightarrow 0\end{array}\right\} \begin{array}{l} C_t \rightarrow S_t - K \\ P_t \rightarrow 0\end{array}$$

- OTM CALL, ITM PUT

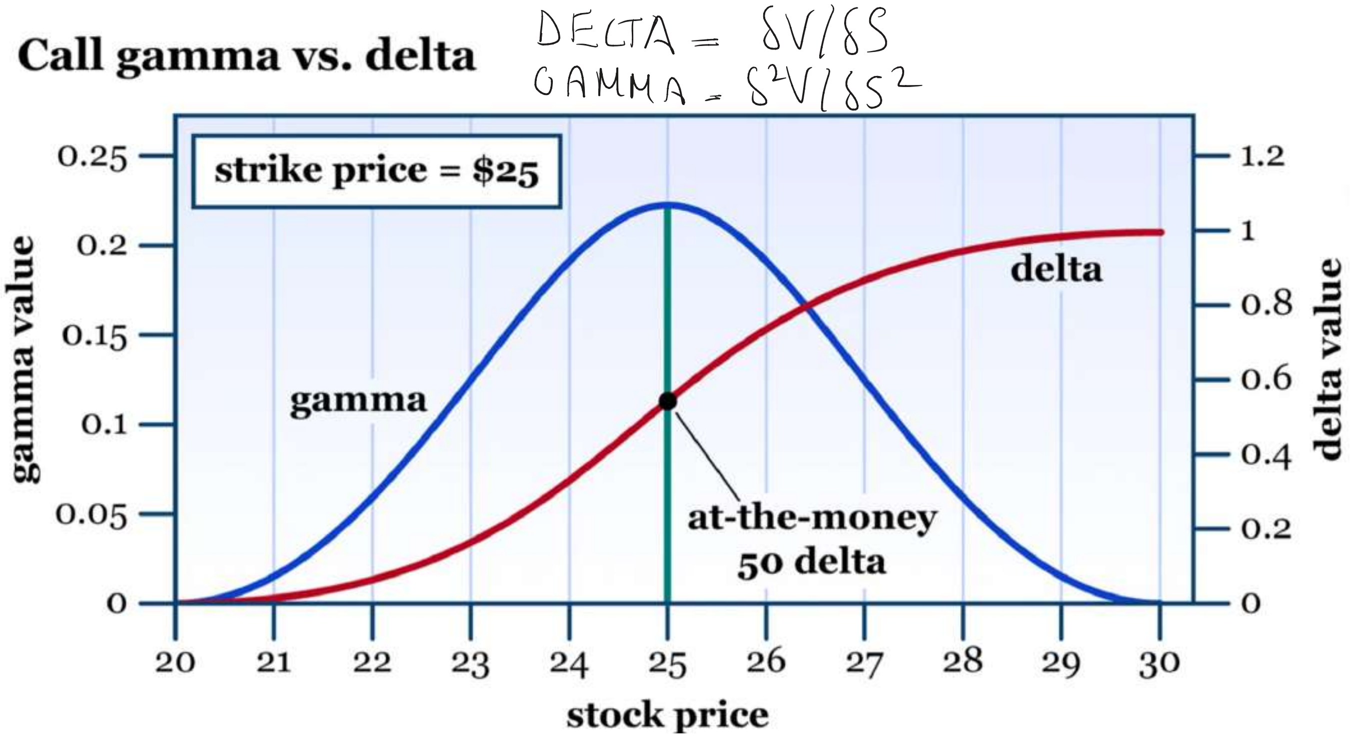
$$\left.\begin{array}{l}\Phi(d_{1t}) \approx \Phi(d_{2t}) \longrightarrow 0 \\ \Phi(-d_{1t}) \approx \Phi(-d_{2t}) \longrightarrow 1\end{array}\right\} \begin{array}{l} C_t \rightarrow 0 \\ P_t \rightarrow K - S_t\end{array}$$

THE GREEKS

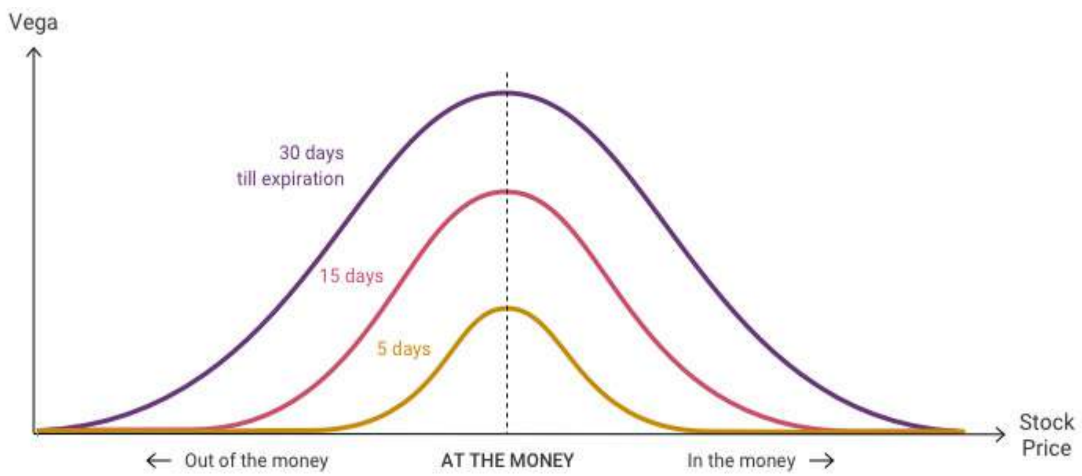
Partial derivatives of a model option price respect to its risk factors

Greek	*Symbol*	*Measures*	*Definition*	
Delta	$\Delta = \frac{\partial V}{\partial S}$	Equity Exposure	Change in option price due to spot	δ
Gamma	$\Gamma = \frac{\partial^2 V}{\partial S^2}$	Payout Convexity	Change in delta due to spot	γ
Theta	$\Theta = \frac{\partial V}{\partial t}$	Time Decay	Change in option price due to time passing	
Vega	$\upsilon = \frac{\partial V}{\partial \sigma}$	Volatility Exposure	Change in option price due to volatility	ν
Rho	$\rho = \frac{\partial V}{\partial r}$	Interest Rate Exposure	Change in option price due to interest rates	
Volga	$\frac{\partial^2 V}{\partial \sigma^2}$	Vol of Vol Exposure	Change in vega due to volatility	
Vanna	$\frac{\partial^2 V}{\partial S \partial \sigma}$	Skew	Change in vega due to spot OR change in delta due to volatility	
Charm	$\frac{\partial^2 V}{\partial S \partial t}$		Change in delta due to time passing	

DELTA = $\delta V / \delta S$
GAMMA = $\delta^2 V / \delta S^2$



VEGA PLOT (change in option price due to change in time)



POSITION GREEKS

A position greek is defined as

$$\boxed{\omega \times N \times \text{standard Greek}}$$

$\omega = \begin{cases} 1 & \text{long position} \\ -1 & \text{short position} \end{cases}$

N = number of shares

For many options on the same underlying the NET POSITION GREEK is the sum of all the individual position greeks

HEDGING OPTIONS

Option market makers balance their book so that the net position Greeks are all close to zero

→ The book is DELTA - GAMMA - VEGA neutral
(would be also for more Greeks)

Neutrality is achieved buying/selling another option on the same underlying

DELTA HEDGING

A delta hedge matches each unit of the OPTION with delta units of the underlying

DELTA NEUTRAL → Value of the portfolio will remain approx unchanged for smalls changes in the underlying

EXAMPLE OF DELTA HEDGING

$\delta = 0{,}5$ and I sell one CALL option to buy 10 shares
→ then the edge is to buy $10 \cdot 0{,}5$ shares

GAMMA AND VEGA HEDGING

Delta changes over time, so the portfolio should be rebalanced continuously. If that's not possible, GAMMA and VEGA positions become relevant.

→ Before delta hedging the position, make sure the net position Gamma and Vega are BOTH ZERO.

(due to linear relation in BS between Gamma and Vega, we need to buy options of DIFFERENT MATURITIES for both edges)

RECALL FOR NEXT TOPIC

Taylor Expansion 2^{ND} order for $f(S,\sigma)$

$$\underbrace{f(S+h_S,\sigma+h_\sigma)-f(S,\sigma)}_{\Delta f} \approx h_S \delta + 0{,}5\, h_S \gamma + h_\sigma V$$

DELTA - VEGA APPROXIMATION

This approximation works well for small changes in S and σ. (S is indeed the most important risk factor for options)

DELTA - GAMMA - VEGA VaR

We take daily changes of S_t and σ, so we can compute the above approximation to obtain daily Δf.

We can therefore compute VaR with historical or MonteCarlo approach.

MINIMUM CAPITAL REQUIREMENTS

Requirement aim to reduce the risk of insolvency by keeping REGULATORY CAPITAL that can't be locked into illiquid assets

MCR must be reserved to cover MARKET, CREDIT and OPERATIONAL risk, each requirement is calculated separately by trading desk and then aggregated.

CALCULATING THE MCR

Introduced by BASEL I standards in 1996:

$$\boxed{MCR_t = \max\left\{ m_1 VaR_t,\ VaR_t^* \right\} + S_t}$$

- VaR_t is the 1% 10-day VaR on day t
- VaR^*_t is the average 1% 10-day VaR over last 60 days
- S_t is an ADD-ON for specific risk factors
- $3 \leq m_1 \leq 4$ is a multiplier determined by backtests

However, in 2016 with Basel III the MCR became:

$$MCR_t = \max\left\{ m_1 VaR_t,\ m_2 VaR_t^* \right\} + S_t$$

- $m_1 \geq 1$

  $1{,}5 \leq m_2 \leq 2$, both determined by a supervisor

THE FUNDAMENTAL REVIEW OF THE TRADING BOOK (FRTB) 2016

The Fundamental Review of the Trading Book changes again the MCR

- h is not fixed anymore to 10, it can increase in case of impairment of liquidity
- EXPECTED SHORTFALL (ES) replace VaR during stressful periods

EXPECTED SHORTFALL (ES)

ES is the expected loss, given that the loss exceeds VaR

(VaR doesn't tell anything about the loss if the VaR exceeds)

$$\boxed{ES_{h,\alpha} = -\mathbb{E}\left[X_h \mid X_h < -VaR_{h,\alpha}\right]}$$

(expressed as % of the portfolio)

where X_h is the discounted h-day return

Generally, ES is found by taking the average of the losses that exceed VaR (using historical data or MonteCarlo)

EXPECTED SHORTFALL IN THE NORMAL VAR MODEL

$$\boxed{ES_{h,\alpha} = \alpha^{-1}\varphi\left(\Phi^{-1}(\alpha)\right)\sigma_h - \mu_h}$$

where φ is the standard normal density function and Φ the distribution function.

BANKING REGULATIONS: BASEL III

VIDEO EXPLANATION OF THE BASEL III REGULAMENTATION CAN BE FOUND

https://youtu.be/KpWBf3s4NpI?si=PweysFys7VdcS32h

GOALS

- Make banks stronger for future financial shocks without causing contagion to other sectors in case of crisis
- Enforce better risk management in all the financial industry (not only banks!) by strengthnening trasparency
- Focus on capital adepuancy (stress testing, fair-value assessments...)

CAPITAL ADEQUACY RATIO (CAR)

$$\boxed{CAR = \frac{\text{TIRE 1 CAPITAL} + \text{TIRE 2 CAPITAL}}{\text{RISK WEIGHTED ASSETS}}}$$

- TIRE 1 CAPITAL: High quality capital: common equity (share capital and retained earnings) and stable resources
- TIRE 2 CAPITAL: Revaluation reserves, hybrid instruments

→ SUM = TOTAL CAPITAL

- RISKY WEIGHTED ASSET: bank's assets weighted on their risk level (low weight → low risk)

CAR must be at least 12,5% ⚠

BACKTESTING VAR MODELS

Backtest are based on VaR exceedances:

SIMPLIFIED STEP-BY-STEP PROCEDURE

- Compute VaR at day t, then record
  $\begin{cases} 1 & \text{if the P\&L exceeds VaR}_t \\ 0 & \text{otherwise} \end{cases}$
- Repeat over historical data

BACKTESTING METHODOLOGY (GENERAL APPROACH)

- Find a candidate portfolio
- Fix estimation period to estimate VaR (at least 250 dd)
- Use rolling-window approach to repeatedly forecast VaR and compare it with realised returns.

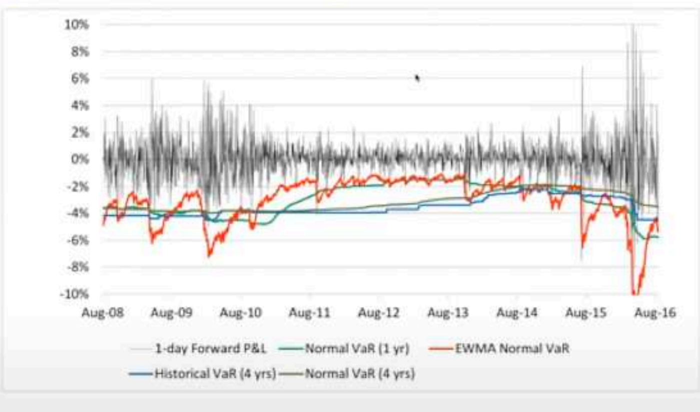


Figure: IBEX 35 1% 1-day VaR & Forward Daily P&L

$$\boxed{I_{\alpha,t} = \begin{cases} 1 & \text{if } X_{t+1} < -VaR_{h,\alpha,t} \\ \emptyset & \text{otherwise} \end{cases}}$$

(indicator function)

If the model captures risk accurately:

$$\boxed{I_\alpha \sim \text{Bernoulli}(\alpha)}$$

So we can define

$$\boxed{S_{n,\alpha} = \sum_t I_{\alpha,t}} \sim \text{BINOMIAL DISTRIBUTION}$$

Under this assumption:

$$\mathbb{E}[S_{n,\alpha}] = n \cdot \alpha \qquad \mathbb{V}[S_{n,\alpha}] = n\alpha(1-\alpha)$$

And a 2-sided 95% interval level for $S_{n,\alpha}$ for n large:

$$\left[n\alpha - 1{,}96\sqrt{n\alpha(1-\alpha)},\ n\alpha + 1{,}96\sqrt{n\alpha(1-\alpha)} \right]$$

If we denote $\alpha = p$ so $\mathbb{E}[S_{n,p}] = np$, $Var[S_{n,p}] = np(1-p)$
we can generalize the above interval formula as:

$$\boxed{\left[\underbrace{np}_{\mathbb{E}(S_{n,p})} - z_{\frac{\alpha}{2}} \sqrt{\underbrace{np(1-p)}_{\mathbb{V}(S_{n,p})}},\ \underbrace{np}_{\mathbb{E}(S_{n,p})} + z_{\frac{\alpha}{2}} \underbrace{\sqrt{np(1-p)}}_{\mathbb{V}(S_{n,p})} \right]}$$

COVERAGE TESTS

Kupiec and Christoffersen created test wheter the i.i.d Bernoulli process has costant success probability α

$$\boxed{LR_{uc} = \frac{\pi_{exp}^{n_1}(1-\pi_{exp})^{n_0}}{\pi_{obs}^{n_1}(1-\pi_{obs})^{n_0}}}$$

test statistic

π_{exp}: expected proportion of exceedances
π_{obs}: observed proportion of exceedances
n_1: number of exceedances
$n_0 = n - n_1$ where n is the sample size

The asymptotic distribution of $-2\ln(LR_{uc})$ is CHI-SQUARED with 1 dof

So, practical steps are

- Compute VaR, backtest and find π_{OBS}, π_{EXP}, n_0, n_1
- Calculate LR_{uc} and then $-2\ln(LR_{uc})$
- Find closes X^2 value to $-2\ln(LR_{uc})$

NULL HYPOTESIS = "The VaR model is accurate"

Reject the VaR model is accurate at $x\%$ confidence if

$$\underbrace{-2\ln(LR_{uc})}_{\text{statistic test}} > \underbrace{X^2_x}_{\text{value of chi-squared distribution with } x\% \text{ significance level}}$$

Don't reject the VaR model is accurate at $y\%$ if

$$-2\ln(LR_{uc}) < X^2_y$$

INDEPENDENCE COVERAGE TEST

Christoffersen proposed a test for independence of the exceedances

$$\boxed{LR_{IND} = \frac{\pi_{OBS}^{n_1}(1-\pi_{OBS})^{n_0}}{\pi_{01}^{n_{01}}(1-\pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}(1-\pi_{11})^{n_{10}}}}$$

$$\pi_{01} = \frac{n_{01}}{(n_{00}+n_{01})} \qquad \pi_{11} = \frac{n_{11}}{(n_{10}+n_{11})}$$

n_{ij} is the number of indicators i followed by indicator j

$$\underbrace{11110}_{n_{10}=3}$$

The asymptotic distribution of $-2\ln(LR_{IND})$ is CHI-SQUARED with 1 dof.

CONDITIONAL COVERAGE TEST

$$LR_{cc} = \frac{\pi_{exp}^{n_1}(1-\pi_{exp})^{n_0}}{\pi_{01}^{n_{01}}(1-\pi_{01})^{n_{00}}\,\pi_{11}^{n_{11}}(1-\pi_{11})^{n_{10}}}$$

The asymptotic distribution of $-2\ln(LR_{cc})$ is CHI-SQUARED 2 dof.

The conditional coverage Test evaluates wheter a predictive model is WELL-CALIBRATED, considering conditioning info (market conditions, volatility, asset Types ecc ecc)

⟶ If The Test fails, The VaR prediction model may need adjustment (incorporating volatility clustering example)

STRESS TESTING AND SCENARIO ANALYSIS

Examples of Hypothetical Scenarios

- Parallel shift in a yield curve of ±100 basis points
- Linear tilt in a yield curve of ±25 basis points
- Parallel change in credit spreads of ±20 basis points
- Stock index return of ±10%
- Return of ±6% on a major currency pair, or of ±20% for a minor currency against another currency
- Relative change in volatility of ±20%

SIX SIGMA EVENTS

Suppose a portfolio exposed To K risk factors whose returns are $X_1 \dots X_K$, The P&L can be denoted as

$$P\&L = f(X_1 \dots X_K)$$

assume also The K risk factors have mean $\hat{\mu}_i$ and standard deviation $\hat{\sigma}_i$

The SIX-SIGMA LOSS (so-called worst scenario) is defined as:

$$f(\hat{\mu}_1 \pm 6\hat{\sigma}_1, \hat{\mu}_2 \pm 6\hat{\sigma}_2, \dots, \hat{\mu}_K \pm 6\hat{\sigma}_K)$$

FACTOR-PUSH METHODOLOGY

$\pm$ are chosen independently for each risk factor to MAXIMISE the loss

SYSTEMATIC EQUITY AND FX VaR

International exposures include FX rates as RISK FACTORS
An investor is exposed to both the local market (ex: S&P500) and the FX rate ($/€ for a European)

⚠ The FX Beta β is always 1!
(currency fluctuations are PROPORTIONAL to the amount invested)

EXAMPLE / EXERCISE

A US investor buys 2$m of UK stocks (EQUITY $\beta = 1{,}5$)

$\sigma_{FTSE100} = 0{,}15$ (equity index)

$\sigma_{\$/£} = 0{,}20$ (FX rate)

$\rho_{FTSE100,\ \$/£} = 0{,}3$ (correlation)

compute the 1% 10-day systematic VaR in $

SOLUTION

$$\hat{Y} = \beta X_1 + X_2 = 1{,}5\, X_1 + X_2$$

$X_1 =$ return of FTSE100
$X_2 =$ return of $/£ ($\beta_{\$/£} = 1$!)

The 10-days std are:

$$\sigma_{FTSE100-10} = \frac{0{,}15}{\sqrt{25}} = 0{,}03$$

$$\sigma_{\$/£-10} = \frac{0{,}20}{\sqrt{25}} = 0{,}04$$

(since $\sigma_{FTSE100}$ volatility is computed on 250 dd, we have to scale by 25 to obtain σ_{10})

The systematic $V(Y)$ is

$$V(Y) = \beta^2 V(X) + V(Y) + 2cov(X,Y)$$

$$\sigma = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} \rightarrow Cov(X,Y) = \sigma\sqrt{V(X)V(Y)}$$

$$= (1{,}5)^2(0{,}03)^2 + (0{,}04)^2 + 2(1{,}5)(0{,}3)(0{,}03)(0{,}04) = 0{,}00471$$

So the 1% 10-days systematic VaR is:

$$1\%\,VaR_{10} = \underbrace{2,3263}_{Z\alpha} \cdot \underbrace{\sqrt{0,00471}}_{\sigma} = 15,9\%$$

Decompose systematic Var into
- EQUITY VAR
- FX VAR

The Equity β is 1,5, so

$$\text{EQUITY VAR}_{1\%,10} = \beta \cdot Z_{\frac{\alpha}{2}} \cdot \sigma_{FTSE100} = 1,5 \cdot 2,3263 \cdot 0,03 = 10,4\%$$

The FX Var is:

$$\text{FX VaR}_{1\%,10} = 1 \cdot Z_{\frac{\alpha}{2}} \cdot \sigma_{FX} = 2,3263 \cdot 0,04 = 9,3\%$$

FX VaR + EQUITY VaR = 19,77 % > TOTAL SYSTEMATIC VAR!

NORMAL LINEAR EQUITY AND FX VAR WITH MULTIPLE RISK FACTORS

$$\boxed{\hat{Y} = \sum^{m}\hat{\beta}_i X_i = (\hat{\beta}_1 X_1 + \dots \hat{\beta}_m X_m) + (X_{m+1} + \dots X_n)}$$

m equity indices, m-1 FX risk factor (or n-m) assuming the investor holds 1 domestic stock

$$V[\hat{Y}] = \hat{\beta}' V \beta$$

where $\hat{\beta} = [\beta_1 \dots \hat{\beta}_m, 1 \dots 1] = [\beta_{EQUITY}, 1 \dots 1]$

DECOMPOSITION IN EQUITY AND FX VAR

Assuming Y, X_i daily returns

$$\text{SYSTEMATIC VAR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{h\,\beta^T V \beta}$$

$$\text{EQUITY VAR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{h\,\beta^T_{EQUITY} V_{EQ} \beta_{EQUITY}}$$

$$\text{FX VAR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{h\,\mathbb{1}\,V_{FX}\,\mathbb{1}}$$

V_{EQ}, V_{FX} are equity and FX portion of Cov Matrix